

Protected gates for topological quantum field theories

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Abstract

We give restrictions on the locality-preserving unitary automorphisms U , which are protected gates, for topologically ordered systems. For arbitrary anyon models, we show that such unitaries only generate a finite group, and hence do not provide universality. For abelian anyon models, we find that the logical action of U is contained in a proper subgroup of the generalized Clifford group. In the case $D(\mathbb{Z}_2)$, which describes Kitaev's toric code, this represents a tightening of statement previously obtained within the stabilizer framework in [10]. For non-abelian models, we find that such automorphisms are very limited: for example, there is no non-trivial gate for Fibonacci anyons. For Ising anyons, protected gates are elements of the Pauli group. These results are derived by relating such automorphisms to symmetries of the underlying anyon model: protected gates realize automorphisms of the Verlinde algebra. We additionally use the compatibility with basis changes to characterize the logical action.

Contents

1	Introduction	2
2	Abelian anyon models	6
2.1	Basic definitions and postulates for string-operators	7
2.2	The generalized Pauli and Clifford groups	10
2.3	Restrictions on protected gates for abelian anyon models	12
3	TQFTs: background	14
3.1	The Verlinde algebra	14
3.2	String-like operators and relations	15
3.3	Bases of the Hilbert space \mathcal{H}_Σ	17
3.4	Open surfaces: labeled boundaries	18
3.5	The gluing axiom	20

4	Locality-preserving automorphisms for TQFTs	22
4.1	Locality-preserving unitaries and symmetries of anyon model	22
4.2	A local constraint from a simple closed loop	22
4.3	A global constraint: DAP-decompositions, fusion rules and the gluing axiom . .	24
4.4	Additional global constraints: DAP-decompositions and basis changes	26
4.4.1	Determining phases for the torus	26
4.4.2	Determining phases for the four-punctured sphere: fixed boundary labels	27
4.4.3	Determining phases for the four-punctured sphere in general	28
4.4.4	Localization of phases for higher-genus surfaces	29
5	Examples	31
5.1	The $D(\mathbb{Z}_2)$ model revisited	32
5.2	The Fibonacci model	33
5.2.1	On the torus	34
5.2.2	Fibonacci on the 4-punctured sphere	35
5.2.3	M Fibonacci anyons: global constraints from fusion and gluing	36
5.2.4	Localization of internal phases for Fibonacci	37
5.3	The Ising model	38
5.3.1	Ising on the 4-punctured sphere	38
5.3.2	M Ising anyons: global constraints from fusion and gluing	40
5.3.3	Localization of internal phases for Ising	40

1 Introduction

In order to reliably compute, it is necessary to protect information against noise. For quantum computations, this is particularly challenging because noise in the form of decoherence threatens the very quantum nature of the process. Adding redundancy by encoding information into a quantum error-correcting code is a natural, conceptually appealing approach towards building noise-resilient scalable computers based on imperfect hardware.

Among the known quantum error-correcting codes, the class of so-called topological codes stands out. Examples in 2D include the toric code and quantum double models [23], the surface codes [8], the 2D color codes [4], variants of these codes [3, 15] and the Levin-Wen model [28]. In 3D, known examples are Bombin and Martin-Delgado’s 3D color code [6], as well as Haah’s [18] and Michnicki’s [32] models. These codes are attractive for a number of reasons: their code space is topologically protected, meaning that small local deformations or locally acting noise do not affect encoded information. The degree of this protection (measured in information-theoretic notions in terms of code distance, and manifesting itself in physical properties such as gap stability) scales with the system size: in other words, robustness essentially reduces to the question of scalability. Finally, the code space of a topological code is the degenerate ground space of a geometrically local Hamiltonian: this means that syndrome information can be extracted by local measurements, an important feature for actual realizations. Furthermore, this implies that a topological code is essentially a phase of a many-body system and can be characterized in terms of its particle content, their statistics, and the quantum field theory emerging in the continuum limit. In particular, this provides a description of such systems which captures all universal features, independently of microscopic details.

While quantum error-correcting codes can provide the necessary protection of information against noise, a further requirement for quantum computation is the ability to execute gates in a robust manner. Again, topological codes stand out: they usually provide certain intrinsic mechanisms for executing gates in a robust way. More precisely, there are sequences of local code deformations, under which the information stays encoded in a code with macroscopic distance, but undergoes some unitary transformation. In principle, this provides a robust implementation of computations by sequences of local, and hence, potentially experimentally realizable actions. In the case of $2D$ -topological codes described by topological quantum field theories, this corresponds to adiabatic movement (braiding) of quasi-particle excitations (also called anyons).

Unfortunately, as is well known, braiding (by which we mean the movement either around each other or more generally around non-trivial loops) of anyons does not always give rise to a universal gate set. Rather, the set of gates is model-dependent: braiding of $D(\mathbb{Z}_2)$ -anyons generates only global phases on the sphere, and elements of the Pauli group on non-zero genus surfaces. Braiding of Ising anyons gives Clifford gates, whereas braiding of Fibonacci anyons generates a dense subgroup of the set of unitaries (and is therefore universal within suitable subspaces of the code space). In other words, braiding alone, without additional tricks such as magic state distillation [9] (which has a large overhead [14]), is not in general sufficient to provide universal fault-tolerant computation; unfortunately, the known systems with universal braiding behavior are of a rather complex nature, requiring e.g., 12-body interactions among spins [28]. Even ignoring the question of universality, the use of braiding has some potentially significant drawbacks: in general (especially for non-abelian anyons), it requires an amount of time which scales with the system size (or code distance) to execute a single logical gate (Mathematically, this is reflected by the fact that string-operators cannot be implemented in finite depth.) This implies that error-correction steps will be necessary even during the execution of such a gate, which may pose an additional technological challenge, for example, if the intermediate topologies are different.

Given the limitations of braiding, it is natural to look for other mechanisms for implementing robust gates in topological codes. For stabilizer quantum codes, the notion of transversal gates has traditionally been used almost synonymously with fault-tolerant gates: their key feature is the fact that they do not propagate physical errors. More generally, for topological stabilizer codes, we can consider logical gates implementable by constant-depth quantum circuits as a proxy for robust gates: they can increase the weight of a physical error only by a constant, and are thus sufficiently robust when combined with suitable error-correction gadgets. Note that finite-depth local circuits represent a much broader class than transversal gates.

Gate restrictions on transversal, as well as constant-depth local circuits have been obtained for stabilizer and more general codes. Eastin and Knill [12] argued that for any code, transversal gates can only generate a finite group and therefore do not provide universality. Bravyi and König [10] consider the group of logical gates that may be implemented by such constant-depth local circuits on geometrically local topological stabilizer codes. They found that such gates are contained in \mathcal{P}_D , the D -th level of the Clifford hierarchy, where D is the spatial dimension in which the stabilizer code is geometrically local.

In this work, we characterize the set of gates implementable by a locality-preserving unitary in a system described by a 2D TQFT. By doing so, we both specialize and generalize the results of [10]: we restrict our attention to dimension 2, but go beyond the set of local stabilizer codes in two significant ways.

First, we obtain statements which are independent of the particular realization (e.g., the toric code model) but are instead phrased in terms of the TQFT (i.e., the anyon model describing the system). In this way, we obtain a characterization which holds for a gapped phase of matter, rather than just for a particular code representing that phase. On a conceptual level, this is similar in spirit to the work of [13], where statements on the computational power for measurement-based quantum computation were obtained that hold throughout a certain phase. Here we use the term phase loosely – we say that two systems are in the same phase if they have the same particle content. To avoid having to make any direct reference to an underlying lattice model, we replace the notion of a constant-depth local circuit by the more general notion of a locality-preserving unitary: this is a unitary operation which maps local operators to local ones.

Second, our results and techniques also apply to non-abelian anyon models (whereas stabilizer codes only realize certain abelian models). In particular, we obtain statements that can be applied, e.g., to the Levin-Wen models [28], as well as chiral phases. Our approach relates locality-preserving unitaries to symmetries of the underlying anyon model; this imposes constraints on the allowed operations. We consider the Fibonacci and Ising models as paradigmatic examples and find that there are no non-trivial gates in the former, and only Pauli operations in the latter case. Our focus on these anyons models is for concreteness only, but our methods and conclusions apply more generally. Our observations are summarized in the following table.

Model	Braiding contained in	locality-preserving unitaries contained in
$D(\mathbb{Z}_2)$	Pauli group	restricted Clifford group
abelian anyon model	generalized Pauli group	generalized Clifford group
Fibonacci model	universal	global phase (trivial)
Ising model	Clifford group	Pauli group
arbitrary anyon model	model-dependent	finite group

Table 1: Braiding is usually considered within a disc-like region. However, the fusion space for abelian anyons in this case is trivial. Braiding abelian anyons on a manifold which supports non-contractible loops can realize non-trivial Pauli gates and their generalization on the code space. Populating the rightmost column of this table is the main result of this work. Our results suggest a trade-off between the computational power of braiding and that of gates implementable by locality-preserving unitaries.

Finally, let us comment on limitations, as well as open problems arising from our work. The first and most obvious one is the dimensionality of the systems under consideration: our methods apply only to $2D$ TQFTs. The mathematics of higher-dimensional TQFTs is less developed, and currently an active research area (see e.g., [26]). While the techniques of [10], which have recently been significantly strengthened by Pastawski and Yoshida [34], also apply to higher-dimensional codes (such as Haah’s), they are restricted to the stabilizer formalism (but importantly, [34] also obtain statements for subsystem codes). Obtaining non-abelian analogues of our results in higher dimensions appears to be a challenging research problem. A full characterization of the case $D = 3$ is particularly desirable from a technological viewpoint.

Even in $2D$, there are obvious limitations of our results: the systems we consider are essentially “homogenous” lattices with anyonic excitations in the bulk. We are not considering defect lines, or condensation of anyons at boundaries; for example, our discussion excludes the quantum double models constructed in [2], which have domain walls constructed from condensation at boundaries using the folding trick. Again, we expect that obtaining statements on protected gates for these models requires additional technology in the form of more refined categorical notions, as discussed by Kitaev and Kong [25]. Also, although we identify possible locality preserving logical unitaries, our arguments do not show that these can necessarily be realized, either in general TQFTs or in specific models that realize TQFTs. Lastly, our work is based on the (physically motivated) assumption that a TQFT description is possible and the underlying data is given. For a concrete lattice model of interacting spins, the problem of identifying this description (or associated invariants [22, 29, 19]), as well as constructing the relevant string-operators (as has been done for quantum double models [23, 5] as well as the Levin-Wen models [28]), is a problem in its own right.

Rough statement of problem

Our results concern families of systems defined on any 2-dimensional orientable manifold (surface) Σ . Typically, such a family is defined in terms of some local physical degrees of freedom (spins) associated with sites of a lattice embedded in Σ . We refer to the joint Hilbert space $\mathcal{H}_{\text{phys},\Sigma}$ of these spins as the ‘physical’ Hilbert space. The Hamiltonian H_Σ on $\mathcal{H}_{\text{phys},\Sigma}$ is local, i.e., it consists only of interactions between “neighbors” within constant-diameter regions on the lattice. More generally, assuming a suitable metric on Σ is chosen, we may define locality in terms of the distance measure on Σ .

We are interested in the ground space \mathcal{H}_Σ of H_Σ . For a topologically ordered system, this ground space is degenerate with dimension growing exponentially with the genus of Σ , and is therefore suitable for storing and manipulating quantum information. We will give a detailed description of this space below (see Section 3); it has preferred basis consisting of labelings associated with some set \mathbb{A} . This is a finite set characterizing all distinct types of anyonic quasiparticle excitations of H_Σ in the relevant low energy sector of $\mathcal{H}_{\text{phys},\Sigma}$.

Importantly, the form of \mathcal{H}_Σ is independent of the microscopic details (in the definition of H_Σ): it is fully determined by the associated TQFT. In mathematical terms, it can be described in terms of the data of a modular tensor category, which also describes fusion, braiding and twists of the anyons. We will refer to \mathcal{H}_Σ as the TQFT Hilbert space.

The significance of \mathcal{H}_Σ is that it is protected: local observables can not distinguish between states belonging to \mathcal{H}_Σ . This implies that \mathcal{H}_Σ is an error-correcting code with the property that local regions are correctable: any operator supported in a small region which preserves the code space must act trivially on it (otherwise it could be used to distinguish between ground states).

To compute fault-tolerantly, one would like to operate on information encoded in the code space \mathcal{H}_Σ by acting with a unitary $U : \mathcal{H}_{\text{phys},\Sigma} \rightarrow \mathcal{H}_{\text{phys},\Sigma}$ on the physical degrees of freedom. There are a number of features that are desirable for such a unitary to be useful – physical realizability being an obvious one. For fault-tolerance, two conditions are particularly natural:

- (i) the unitary U should preserve the code space, $U\mathcal{H}_\Sigma = \mathcal{H}_\Sigma$ so that the information stays encoded. We call a unitary U with this property an automorphism of the code and denote

its restriction to \mathcal{H}_Σ by $[U] : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma$. The action $[U]$ defines the logical operation or gate that U realizes.

- (ii) typical errors remain correctable under the application of the unitary U . In the context of topological codes, which correct sufficiently local errors, and where a local error model is usually assumed, this condition is satisfied if U does not significantly change the locality properties of an operator: if an operator X has support on a region $\mathcal{R} \subset \Sigma$, then the support of UXU^\dagger is contained within a constant-size neighborhood of \mathcal{R} . We call such a unitary a locality-preserving unitary.

We call a unitary U satisfying (i) and (ii) a locality-preserving automorphism of the code (or simply a topologically protected gate). Our goal is to characterize the set of logical operations that have the form $[U]$ for some locality-preserving¹ automorphism U . For example, if \mathcal{H}_Σ is a topologically ordered subspace of $\mathcal{H}_{\text{phys},\Sigma}$, the Hilbert space of a spin lattice, then (ii) is satisfied if U is a constant-depth local circuit. Another important example is the constant-time evolution $U = \mathcal{T} \exp[-i \int dt H(t)]$ of a system through a bounded-strength geometrically-local Hamiltonian $H(t)$. Here, Lieb-Robinson bounds [31, 7] provide quantitative statements on how the resulting unitary may be exponentially well approximated by a locality-preserving unitary. This is relevant since it describes the time evolution of a physical systems and can also be used to model adiabatic transformations of the Hamiltonian [11].

From a computational point of view, the group

$$\langle \{[U] \mid U \text{ locality-preserving automorphism} \} \rangle$$

generated by such gates is of particular interest: it determines the computational power of gates that are implementable fault-tolerantly.

Outline

In Section 2, we discuss abelian anyon models, starting with a discussion of string-operators and their properties. We then derive the Clifford group characterization of protected gates in abelian models; this follows closely the argument in [10], but goes further. The methods are largely independent of the more general approach for non-abelian models, but may serve as a useful preparation for the latter. In Section 3, we provide a brief introduction to the relevant concepts of TQFTs. We then derive our main results on the characterization of protected gates in Section 4. In Section 5, we apply our results to particular models, deriving in particular our characterizations for Ising and Fibonacci anyons.

2 Abelian anyon models

Our goal in this section is to characterize topologically protected gates in general abelian anyon models. We begin by examining properties of string-operators in such models, relating them

¹ As a side remark, we mention that our terminology is chosen with spin lattices in mind. However, the notion of locality-preservation can be relaxed. As will become obvious below, our results apply more generally to the set of *homology-preserving* automorphisms U . The latter can be defined as follows: if the support of an operator X is contained in a region $\mathcal{R} \subset \Sigma$ which deformation retracts to a closed curve C , then the support of UXU^\dagger must be contained in a region $\mathcal{R}' \subset \Sigma$ which deformation retracts to a curve C' in the same homology class as C . For example, for a translation-invariant system, translating by a possibly extensive amount realizes such a homology-preserving (but not locality-preserving) automorphism.

to a few basic postulates (Section 2.1). The relevant Pauli and Clifford groups will be defined in Section 2.2. Finally, in Section 2.3, we give our main statement for abelian anyon models.

Throughout this section, we will restrict our attention to closed 2-manifolds Σ . These are characterized by their genus g . Fig. 1 illustrates the 3-handled torus Σ_g corresponding to $g = 3$. (We will discuss manifolds with boundaries later in the general context of non-abelian anyons in Section 3.4.)

Associated with the manifold is a physical Hilbert space $\mathcal{H}_{\text{phys},\Sigma}$ of physical degrees of freedom. We take the code space to be the ground space $\mathcal{H}_\Sigma \subset \mathcal{H}_{\text{phys},\Sigma}$ of a local Hamiltonian H_Σ , for a topologically ordered system with abelian anyons. \mathcal{H}_Σ is an error-correcting code with the property that local regions are correctable: any operator supported in a small region which preserves the code space must act trivially on it.

2.1 Basic definitions and postulates for string-operators

An abelian anyon model, with anyon types given by a finite set \mathbb{A} , is equipped with a commutative and associative fusion operator $\times : (\mathbb{A}, \mathbb{A}) \rightarrow \mathbb{A}$ for which 1 is the neutral element. This means that any two particles a and b fuse to a unique particle $c = a \times b$, with $1 \in \mathbb{A}$ being the only particle satisfying $1 \times a = a$ for all $a \in \mathbb{A}$. Furthermore, every particle $a \in \mathbb{A}$ will have a unique antiparticle denoted by $\bar{a} \in \mathbb{A}$ such that $a \times \bar{a} = 1$. Note that these conditions imply that (\mathbb{A}, \times) forms a finite abelian group, and by the fundamental theorem of finitely generated abelian groups, is of the form $(\mathbb{A}, \times) = (\mathbb{Z}_{N_1} \oplus \mathbb{Z}_{N_2} \oplus \dots \oplus \mathbb{Z}_{N_r}, +)$ with each N_j a power of a prime.

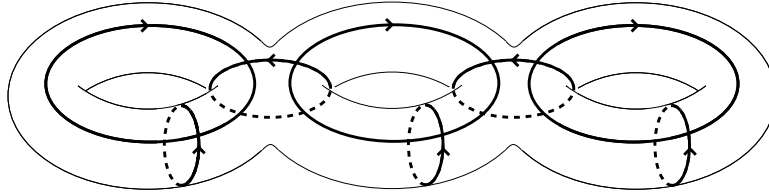


Figure 1: A canonical set of $3g - 1$ generators of the mapping class group of the surface Σ_g can be specified in terms of a set $\mathcal{G} = \{C_j\}_{j=1}^{3g-1}$ of loops (each associated with a Dehn twist). Dragging an anyon a around such loop $C : [0, 1] \rightarrow \Sigma$ and fusing to the vacuum implements an undetectable operator $F_a(C)$; homologically non-trivial loops realize logical operations. The full algebra of logical operators is generated by the set of operators $\{F_a(C)\}_{a \in \mathbb{A}, C \in \mathcal{G}}$. However, these operators are generally not independent.

Consider a “string” or “ribbon” operator $F_a(C)$ implemented by creating a particle-antiparticle pair (a, \bar{a}) of anyons in some local region, and then “dragging” the particle around some non-contractible loop $C : [0, 1] \rightarrow \Sigma$ until it returns to the location $C(1) = C(0)$ of the antiparticle (which we leave stationary) and annihilates with it. This annihilation occurs with probability one in abelian anyon models, i.e., the corresponding operator $F_a(C)$ is unitary (however, this is not the case in general). Note that reversing the direction of C , i.e., considering $C^{-1}(t) \equiv C(1 - t)$, is equivalent to exchanging the particle with its antiparticle, i.e., $F_a(C) = F_{\bar{a}}(C^{-1})$.

We denote the restriction of $F_a(C)$ to the code space \mathcal{H}_Σ by $[F_a(C)]$; since \mathcal{H}_Σ is preserved, $[F_a(C)]$ is a logical unitary in abelian models.

Example 2.1 ($D(G)$ and Kitaev's toric code). As an example, consider a model described by the quantum double $D(G)$ of a finite group G , for which Kitaev has constructed a lattice model [23]. In the case where G is abelian, we have $D(G) \cong G \times G$, i.e., the particles and fusion rules are simply given by the product group $\mathbb{A} = G \times G$.

Specializing to $G = \mathbb{Z}_2$ gives the particles commonly denoted by $1 = (0,0)$ (vacuum), $m = (1,0)$, $e = (0,1)$ and $\epsilon = m \times e = (1,1)$. For the toric code model, the associated ribbon operators are

$$F_1(C) = \text{id} \quad F_e(C) = \bar{X}(C) \quad F_m(C) = \bar{Z}(C) \quad F_\epsilon(C) = \bar{X}(C)\bar{Z}(C) ,$$

where $\bar{X}(C) = \otimes_{j \in \partial_+ C} X_j$ and $\bar{Z}(C) = \otimes_{j \in \partial_- C} Z_j$ are appropriate tensor products of Pauli- X and Pauli- Z -operators along C (as specified in [23]).

Specializing to $G = \mathbb{Z}_N$, with $\omega_N = \exp(2\pi i/N)$ and generalized N -dit Pauli operators X and Z (and their inverses), defined by their action

$$X|j\rangle = |j+1 \bmod N\rangle \quad Z|j\rangle = \omega_N^j |j\rangle$$

on computational basis states $\{|j\rangle\}_{j=0,\dots,N-1}$, we can consider a similar model (the \mathbb{Z}_N -toric code) with generalized ribbon operators. Here

$$F_{(a,a')}(C) = \bar{X}(C)^a \bar{Z}(C)^{a'} ,$$

where $\bar{X}(C)$ is a tensor product of Pauli- X and its inverse depending on the orientation of the underlying lattice, and similarly for $\bar{Z}(C)$.

It is easy to check that operators associated with the same loop commute, i.e.,

$$[F_{(a,a')}(C), F_{(b,b')}(C)] = 0 , \tag{1}$$

and since $Z^a X^b = \omega_N^{ab} X^b Z^a$, we get the commutation relation

$$F_{(a,a')}(C_1) F_{(b,b')}(C_2) = \omega_N^{ab' - a'b} F_{(b,b')}(C_2) F_{(a,a')}(C_1) \tag{2}$$

for any two strings C_1, C_2 intersecting once.

Returning to the general case where $\mathbb{A} = \mathbb{Z}_{N_1} \oplus \dots \oplus \mathbb{Z}_{N_r}$, we will now argue that identities analogous to (1) and (2) hold in general abelian anyon models under very general assumptions. We express these as postulates; they can be seen as a subset of the isotopy-invariant calculus of labeled ribbon graphs associated with the underlying category (see e.g., [16] for a discussion of the latter) and are assumed to be valid for all anyon models considered in this work.

Postulate 2.2 (Completeness of string-operators). Consider an operator U with support in some region \mathcal{R} which preserves the code space \mathcal{H}_Σ . Then its action on the code space is equivalent to that of a linear combination of products of operators of the form $F_a(C)$, where $C : [0, 1] \rightarrow \mathcal{R}$ is supported in \mathcal{R} . That is, we have

$$[U] = \sum_j \beta_j \prod_k [F_{a_{j,k}}(C_{j,k})] .$$

This postulate essentially means that, as far as the logical action is concerned, we may think of $[U]$ as a linear combination of products of string operators. Such products $F_{a_m}(C_m) \cdots F_{a_1}(C_1)$ can conveniently be thought of as ‘labeled’ loop gases embedded in the three-manifold $\Sigma \times [0, 1]$, where, for some $0 < t_1 < \cdots < t_m < 1$, the operator $F_{a_j}(C_j)$ is applied at ‘time’ t_j (and hence a labeled loop is embedded in the slice $\Sigma \times \{t_j\}$). Diagrammatically, one represents such a product by the projection onto Σ with crossings representing temporal order, as in

$$F_{a_2}(C_2)F_{a_1}(C_1) = \begin{array}{c} \text{Diagram: Two overlapping circles. The left circle is labeled } F_{a_2}(C_2) \text{ and the right circle is labeled } F_{a_1}(C_1). \text{ Arrows on the circles indicate a clockwise direction.} \end{array} \quad (3)$$

One may manipulate every term in a linear combination representing U without changing the logical action according to certain local ‘moves’; in particular, the order of application of these moves is irrelevant (a fact formalized by MacLane’s theorem [27]).

For our purposes, we only require the following two ‘local’ moves, which relate two products U and U' of string-operators given by diagrams such as (3). More generally, they may be applied term-by-term to any linear combination if each term contains the same local sub-diagram.

Postulate 2.3 (String deformation). *Suppose U and U' are identical on the complement of some region \mathcal{R} . Assume further that inside \mathcal{R} , both U and U' contain a single string describing the dragging of the same anyon type along a path C and C' , respectively, where C' can be locally deformed into C . Then the logical action of U and U' must be equivalent: $[U] = e^{i\theta}[U']$ for some unimportant phase $e^{i\theta}$ (Fig. 2.3).*

In particular, this postulate implies that if C and C' are two closed homologically equivalent loops and a is an arbitrary anyon label, then the unitaries $F_a(C)$ and $F_{a'}(C)$ realized by “dragging” the specified anyon along C and C' have equivalent logical action on the code space, $[F_a(C)] = e^{i\theta}[F_a(C')]$.

The third postulate we need concerns two anti-parallel strings with the same particle label. Here we present a simplified version pertaining only to abelian anyons.

Postulate 2.4 (Anti-parallel string decoupling). *Suppose U and U' are identical on the complement of some region \mathcal{R} . Assume that inside \mathcal{R} , both U and U' contain a pair of anti-parallel strings describing the dragging of the same type of anyon. Then the logical action of U and U' must be equivalent: $[U] = e^{i\theta}[U']$ for some phase $e^{i\theta}$ (Fig. 2(b)).*

We can easily derive some important consequences of these postulates. It will be useful to introduce the notion of conjugate loops: we say that C and C' are conjugate if they intersect only at one point.

Lemma 2.5 (String-operators in abelian anyon models). *The string-operators satisfy the following commutation relations:*

- (i) *String-operators are invertible with inverse corresponding to the inverse $g^{-1} = \bar{g}$ in the group \mathbb{A} of particle labels up to a phase, i.e.,*

$$[F_g(C)]^{-1} = e^{i\varphi_g}[F_{g^{-1}}(C)] . \quad (4)$$

for any loop C and particle label $g \in \mathbb{A}$. In particular, for the identity element $1 \in \mathbb{A}$, the operators $[F_1(C)]$ is proportional to the identity up to a global phase.

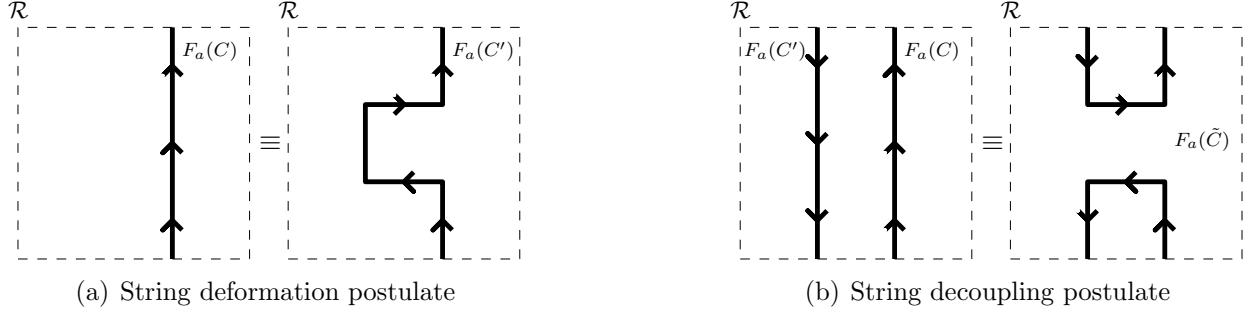


Figure 2: The content of Postulates 2.3 and 2.4: We can deform a line without changing the logical action of the string-operator (Fig. 2(a)). If two anti-parallel lines occur, they can be split as shown in Fig. 2(b).

(ii) For any loop C and particle labels $g, h \in \mathbb{A}$, we have $[F_g(C)][F_h(C)] = [F_h(C)][F_g(C)]$.

(iii) If C and C' are conjugate loops, then the associated string-operators satisfy

$$[F_g(C)][F_h(C')] = e^{i\varphi_{g,h}} [F_h(C')][F_g(C)] . \quad (5)$$

for some phases $\varphi_{g,h}$ depending only on $g, h \in \mathbb{A}$.

In fact, for rational abelian anyon models, the phases appearing in Lemma 2.5 are all roots of unity. More explicitly, if $\mathbb{A} = \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_r}$, and $N = \text{lcm}(N_1, \dots, N_r)$ is the least common multiple, then $e^{i\varphi_g}, e^{i\varphi_{g,h}} \in \langle e^{2\pi i/N} \rangle$, where $\langle e^{2\pi i/N} \rangle$ is the phase subgroup of $U(1)$ consisting of N -th roots of unity.

Proof. Because $F_{g^{-1}}(C) = F_g(C^{-1})$ (inversion of direction), we can apply Postulate 2.4 to the product $F_g(C)F_{g^{-1}}(C)$, obtaining $[F_g(C)F_{g^{-1}}(C)] = e^{i\varphi_g}[F_g(C')]$ for some contractible loop C' . By Postulate 2.3 and the error-correction condition, we must have $[F_g(C)][F_{g^{-1}}(C)] = e^{i\varphi_g}[I]$ and (i) follows.

Because of Postulate 2.3, we may choose a loop C' which is homologically equivalent to C such that $F_h(C')$ and $F_g(C)$ have disjoint support, and $[F_h(C)] = [F_h(C')]$. Property (ii) follows immediately from this.

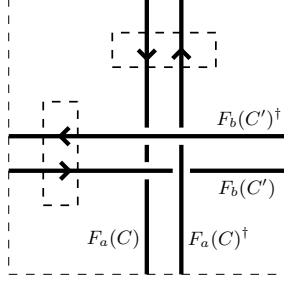
Finally, for the proof of (iii), observe that $[F_g(C)]^{-1}[F_h(C')]^{-1}[F_g(C)][F_h(C')] = e^{i\varphi_{g,h}}[I]$ for some phase $\varphi_{g,h} \in [0, 2\pi)$, since, using (4) and the postulates, this sequence of string operators can be deformed into a contractible link with support on a small region - which can have no non-trivial logical action on the code space (see Fig. 3). \square

2.2 The generalized Pauli and Clifford groups

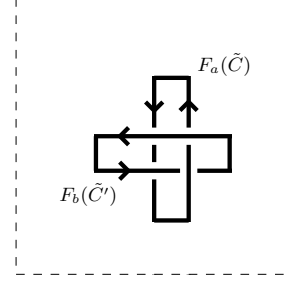
Consider the case where $\mathbb{A} = \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_r}$ and set $N = \text{lcm}(N_1, \dots, N_r)$. We will define the following group associated with the surface Σ .

Definition 2.6 (Pauli group). *Consider a genus- g surface Σ_g and let $\mathcal{G} = \{C_j\}_{j=1}^{3g-1}$ be the loops associated with generators of the mapping class group as in Fig. 1. The Pauli group Pauli_{Σ_g} associated with Σ_g is*

$$\text{Pauli}_{\Sigma_g} := \langle \{ e^{i\varphi}[F_a(C)] \mid e^{i\varphi} \in \langle e^{2\pi i/N} \rangle, a \in \mathbb{A}, C \in \mathcal{G} \} \rangle ,$$



(a) This figure illustrates the loops on a torus corresponding to pair-creation of anyon a and \bar{a} , and dragging \bar{a} around a non-trivial loop C before reannihilation, followed by the same process with an anyon pair b and \bar{b} , but while dragging around the conjugate (intersecting) loop C' , and then undoing each process by dragging in the reverse direction. This is represented as $F_b(C')^\dagger F_a(C)^\dagger F_b(C') F_a(C) = F_{\bar{b}}(C') F_{\bar{a}}(C) F_b(C') F_a(C)$.



(b) The process of Fig. 3(b) can be deformed into a link without changing its logical action. This simplified figure is obtained by applying the string decoupling rule (cf. Fig. 2(b)) to the two dashed squares, and subsequently applying the deformation rule (cf. Fig. 2(a)).

Figure 3: The process defined by the product $F_b(C')^\dagger F_a(C)^\dagger F_b(C') F_a(C)$ can be simplified by applying Postulates 2.3 and 2.4. The resulting link in Fig. 3(b) is supported on some small (and therefore correctable) region, hence the logical action is trivial. This argument underlies both the proof of Lemma 2.5 as well as Theorem 2.11.

i.e., the set of logical operators generated by taking products of string-operators associated with \mathcal{G} , including phases from the subgroup $\langle e^{2\pi i/N} \rangle \subset U(1)$.

According to Lemma 2.5, we can always fix an ordering of the loops and write each element $P \in \text{Pauli}_{\Sigma_g}$ in the standard form

$$P = e^{i\varphi} [F_{a_1}(C_1)] \cdots [F_{a_{3g-1}}(C_{3g-1})] \quad \text{for some } e^{i\varphi} \in \langle e^{2\pi i/N} \rangle, a_j \in \mathbb{A}$$

which shows that the group Pauli_{Σ_g} is finite. Furthermore, since $a^N = 1$ for every $a \in \mathbb{A}$, we conclude that $P^N = e^{i\varphi} [\text{id}]$ is proportional to the identity up to a phase $e^{i\varphi} \in \langle e^{2\pi i/N} \rangle$. That is, every element of the Pauli group Pauli_{Σ} has order dividing N .

Given this definition, we can proceed to give the definition of the Clifford group.

Definition 2.7 (Clifford group). *The Clifford group associated with Σ is the group of logical unitaries*

$$\text{Clifford}_{\Sigma} := \{ e^{i\varphi} [U] \mid [U] \text{Pauli}_{\Sigma} [U]^{-1} \subset \text{Pauli}_{\Sigma}, e^{i\varphi} \in \langle e^{2\pi i/N} \rangle \} .$$

In this definition, $[U]$ is any logical unitary on the code space.

We can define a ‘homology-preserving subgroup’ of Clifford_{Σ} . To do so, we first introduce the following subgroup of Pauli_{Σ} associated with a loop on Σ .

Definition 2.8 (Restricted Pauli group). *Let $C \in \mathcal{G}$ be a single closed loop. We set*

$$\text{Pauli}_{\Sigma_g}(C) := \langle \{ e^{i\varphi} [F_a(C)] \mid e^{i\varphi} \in \langle e^{2\pi i/N} \rangle, a \in \mathbb{A} \} \rangle ,$$

i.e., the subgroup generated by string-operators associated with the loop C .

It is straightforward to check that for any $C \in \mathcal{G}$, the subgroup $\text{Pauli}_{\Sigma_g}(C) \subset \text{Pauli}_{\Sigma_g}$ is normal; furthermore, any $P \in \text{Pauli}_{\Sigma_g}(C)$ has the simple form of a product $P = e^{i\varphi} [F_{a_1}(C)] \cdots [F_{a_r}(C)]$.

Given this definition, we can define a subgroup of Clifford group elements as follows:

Definition 2.9 (Homology-preserving Clifford group). *The homology-preserving Clifford group associated with Σ is the subgroup*

$$\text{Clifford}_{\Sigma}^* := \{e^{i\varphi}[U] \mid [U]\text{Pauli}_{\Sigma}(C)[U]^{-1} \subset \text{Pauli}_{\Sigma}(C) \text{ for all } C \in \mathcal{G}, e^{i\varphi} \in \langle e^{2\pi i/N} \rangle\}.$$

Note that this is a proper subgroup, i.e., $\text{Clifford}_{\Sigma}^* \subsetneq \text{Clifford}_{\Sigma}$, as can be seen from the following example.

Example 2.10. *Consider for example Kitaev's $D(\mathbb{Z}_2)$ -code on a torus Σ_2 (cf. Example 2.1). In this case, there are two inequivalent homologically non-trivial cycles C_1 and C_2 . In the language of stabilizer codes, the logical operators $(\bar{X}_1, \bar{Z}_1) = (F_e(C_1), F_m(C_2))$ and $(\bar{X}_2, \bar{Z}_2) = (F_e(C_2), F_m(C_1))$ are often referred to as the logical Pauli operators associated with the first and second logical qubit, respectively. Consider the logical Hadamard \bar{H}_1 on the first qubit, which acts as*

$$\bar{H}_1 \bar{X}_1 \bar{H}_1^\dagger = \bar{Z}_1 \quad \text{and} \quad \bar{H}_1 \bar{Z}_1 \bar{H}_1^\dagger = \bar{X}_1$$

but leaves \bar{X}_2 and \bar{Z}_2 invariant. Then \bar{H}_1 belongs to the Clifford group, $\bar{H}_1 \in \text{Clifford}_{\Sigma}$. However, $\bar{H}_1 \notin \text{Clifford}_{\Sigma}^$ because \bar{X}_1 and \bar{Z}_1 belong to different homology classes (specified by C_1 and C_2 , respectively).*

2.3 Restrictions on protected gates for abelian anyon models

Now we can state our main result for abelian models:

Theorem 2.11. *For an abelian anyon model, any locality-preserving unitary automorphism U has logical action $[U] \in \text{Clifford}_{\Sigma}^*$.*

The proof of this statement proceeds similarly as in [10]. It is essentially based on a reinterpretation of the cleaning lemma for stabilizer codes: in the context of topological codes, it is substituted by the deformation property of string-operators.

In Section 5.1, we give an alternative proof of the same statement based only on the fusion rules for the special case where $\mathbb{A} = D(\mathbb{Z}_2)$; however, the reasoning following [10] appears to be necessary to establish the statement of Theorem 2.11 for arbitrary abelian anyon models.

Proof. Consider a logical unitary $[F_a(C)]$ implemented by a string operator $F_a(C)$, and the “fattened” string operator $\mathbf{F}_a(C) = U F_a(C) U^\dagger$. Since Pauli_{Σ} spans the full algebra of operators on the code space \mathcal{H}_{Σ} , the logical operator $[\mathbf{F}_a(C)]$ must be in its linear span, i.e., $[\mathbf{F}_a(C)] \in \text{span}(\text{Pauli}_{\Sigma})$. By locality, we know that in fact, $[\mathbf{F}_a(C)] \in \text{span}(\text{Pauli}_{\Sigma}(C))$, that is,

$$[\mathbf{F}_a(C)] = \sum_{c \in \mathbb{A}} \Lambda_{a,c} [F_c(C)] \tag{6}$$

for some coefficients $\{\Lambda_{a,c}\}_c$. We will show that only one of these coefficients can be non-zero, i.e., $[\mathbf{F}_a(C)] \in \text{Pauli}_{\Sigma}(C)$. In particular, since $F_a(C)$ was arbitrary, this implies the claim that $U \in \text{Clifford}_{\Sigma}^*$.

A key element in our proof is the observation that the collection of “fattened” string-operators $\{\mathbf{F}_a(C) := UF_a(C)U^\dagger\}_{a \in \mathbb{A}, C}$ arbitrary satisfies the same postulates (discussed in Section 2.1) as the “usual” string-operators $\{F_a(C)\}_{a, C}$: this is because “fattening” can be done term-wise in products (e.g., we have $[\mathbf{F}_a(C)\mathbf{F}_b(C')] = [U][F_a(C)][F_b(C')][U]^\dagger$) and hence the original postulates apply. Furthermore, can apply the local rules to diagrams involving both fattened- and unfattened strings simultaneously, where every labeled fattened string represents a linear combination such as (6).

This implies that the same argument as used in the proof of Lemma 2.5 applies to any pair of operators $\mathbf{F}_a(C)$ and $F_b(C')$, where C and C' are conjugate, identical, or disjoint (this is obtained by considering $\mathbf{F}_{\bar{a}}(C)F_b(C')\mathbf{F}_a(C)F_b(C')$): we have that

$$[\mathbf{F}_a(C)][F_b(C')] = e^{i\theta_{a,b}}[F_b(C')][\mathbf{F}_a(C)] . \quad (7)$$

for some phase $\theta_{a,b}$.

Together with Lemma 2.5, the constraint (7) suffices to conclude that the rhs. of (6) is indeed proportional to a single string-operator. Indeed, we can rewrite (7) as

$$\sum_{c \in \mathbb{A}} \Lambda_{a,c} ([F_c(C)][F_b(C')] - e^{i\theta_{a,b}}[F_b(C')][F_c(C)]) = 0 .$$

Using the commutation relations from Lemma 2.5 and left-multiplying by $[F_b(C')]^{-1}$, we get

$$\sum_{c \in \mathbb{A}} \Lambda_{a,c} (e^{i\varphi_{c,b}} - e^{i\theta_{a,b}}) [F_c(C)] = 0 .$$

The operators $\{[F_c(C)]\}_c$ are linearly independent, and hence we must have for all $a, c \in \mathbb{A}$

$$\Lambda_{a,c} = 0 \quad \text{or} \quad (\Lambda_{a,c} \neq 0 \text{ and } e^{i\theta_{a,b}} = e^{i\varphi_{c,b}} \text{ for all } b) \quad (8)$$

(since the choice of b was arbitrary and is independent of c). We now argue that this determines all phases $\theta_{a,b}$ in such a way that

$$e^{i\theta_{a,b}} = e^{i\varphi_{c_*,b}} \quad \text{for some } c_* = c_*(a) \in \mathbb{A} . \quad (9)$$

Indeed, let $a, b \in \mathbb{A}$ be arbitrary. Then there must be an element $c_* = c_*(a) \in \mathbb{A}$ such that $\Lambda_{a,c_*} \neq 0$ (since $\mathbf{F}_a(C)$ is a non-zero operator). The claim (9) then follows from (8).

With the phases $e^{i\theta_{a,b}}$ fixed by (9), the action of $[\mathbf{F}_a(C)]$ by conjugation on Pauli group elements is completely determined by (7), i.e., we have

$$[\mathbf{F}_a(C)][F_b(C')] = e^{i\varphi_{c_*(a),b}}[F_b(C')][\mathbf{F}_a(C)] \quad \text{for all } b \in \mathbb{A} . \quad (10)$$

Since the operator $[F_{c_*(a)}(C)]$ obeys the same commutation relation with the operators $F_b(C'), b \in \mathbb{A}$, we conclude that $[\mathbf{F}_a(C)] = e^{i\varphi}[F_{c_*(a)}(C)]$ for some phase $e^{i\varphi}$. But since

$$[\mathbf{F}_a(C)]^N = [U][F_a(C)]^N[U^\dagger] = [\text{id}]$$

because elements of Pauli_Σ have order dividing N , we must have $e^{i\varphi N} = 1$, i.e., $e^{i\varphi} \in \langle e^{2\pi i/N} \rangle$. This implies that $[\mathbf{F}_a(C)] \in \text{Pauli}_\Sigma$ is an element of the Pauli group. Because a and C were arbitrary, this concludes (with (6)) the proof that $U \in \text{Clifford}_\Sigma^*$. \square

3 TQFTs: background

In this section, we provide the necessary background on topological quantum field theories (TQFTs). Our discussion will be rather brief; for a more detailed discussion of topological quantum computation and anyons, we refer to [35]. Following Witten's work [38], TQFTs have been axiomatized by Atiyah [1] based on Segal's work [36] on conformal field theories. Moore and Seiberg [33] derived the relations satisfied by the basic algebraic data of such theories (or more precisely, a modular functor). Here we borrow some of the terminology developed in full generality by Walker [21] (see also [17]). For a thorough treatment of the category-theoretic concepts, we recommend the appendix of [24].

Our focus is on the Hilbert space \mathcal{H}_Σ spanned by the vacuum states of a TQFT defined on the surface Σ . Recall that this is generally a subspace $\mathcal{H}_\Sigma \subset \mathcal{H}_{\text{phys},\Sigma}$ of a Hilbert space of physical degrees of freedom. We are interested in the algebra \mathcal{A}_Σ of operators $X : \mathcal{H}_{\text{phys},\Sigma} \rightarrow \mathcal{H}_{\text{phys},\Sigma}$ which preserve the subspace \mathcal{H}_Σ . We call such an element $X \in \mathcal{A}_\Sigma$ an automorphism and denote by $[X] : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma$ the restriction to \mathcal{H}_Σ . We call X a representative (or realization) of $[X]$. Operators of the form $[X]$, where $X \in \mathcal{A}_\Sigma$, define an associative $*$ -algebra $[\mathcal{A}_\Sigma]$ with unit and multiplication $[X][Y] = [XY]$. The unit element in $[\mathcal{A}_\Sigma]$ is represented by the identity operator id on the whole space $\mathcal{H}_{\text{phys},\Sigma}$.

3.1 The Verlinde algebra

Let \mathbb{A} be the set of particle labels. Let $N_{ab}^c = \dim V_{ab}^c = \dim V_c^{ab}$ be the fusion multiplicity. We will restrict our attention to models where $N_{ab}^c \in \{0, 1\}$ for all $a, b, c \in \mathbb{A}$ for simplicity (our results generalize with only minor modifications) and write $\delta_{abc} = N_{ab}^c$. The *Verlinde algebra* \mathbf{Ver} is the commutative associative $*$ -algebra spanned by elements $\{\mathbf{f}_a\}_{a \in \mathbb{A}}$ satisfying the relations

$$\mathbf{f}_a \mathbf{f}_b = \sum_c N_{ab}^c \mathbf{f}_c \quad \text{and} \quad \mathbf{f}_a^\dagger = \mathbf{f}_{\bar{a}} .$$

Note that $\mathbf{f}_1 = \text{id}$ is the identity element because $N_{a1}^c = N_{1a}^c = \delta_{ac}$.

If braiding is defined, we have $N_{ab}^c = N_{ba}^c$, and \mathbf{Ver} is a finite-dimensional commutative C^* -algebra. Therefore $\mathbf{Ver} \cong \mathbb{C}^{\oplus(\dim \mathbf{Ver})}$ is a direct sum of copies of \mathbb{C} . The fusion multiplicity N_{ab}^c may also be written in terms of the modular S -matrix, whose matrix elements are, in the diagrammatic calculus, given by the Hopf link and the total quantum dimension \mathcal{D} by

$$S_{ab} = \frac{1}{\mathcal{D}} a \bigcirc b .$$

We consider the case where the S -matrix is unitary: here the isomorphism $\mathbf{Ver} \cong \mathbb{C}^{\oplus(\dim \mathbf{Ver})}$ can be made explicit thanks to the *Verlinde formula* [37]

$$N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} S_{\bar{c}x}}{S_{1x}} . \quad (11)$$

(Note that $S_{1x} = d_x / \mathcal{D}$ where $\mathcal{D} = \sqrt{\sum_a d_a^2}$.) For this purpose, we define the elements

$$\mathbf{p}_a = S_{1a} \sum_b \bar{S}_{ba} \mathbf{f}_b \quad \text{for all } a \in \mathbb{A} . \quad (12)$$

The main statement we use is the following:

Proposition 3.1 (Primitive idempotents). *The elements $\{\mathbf{p}_a\}_{a \in \mathbb{A}}$ are the unique complete set of orthogonal minimal idempotents spanning the Verlinde algebra,*

$$\text{Ver} = \bigoplus_a \mathbb{C} \mathbf{p}_a . \quad (13)$$

Furthermore, they satisfy

$$\sum_a \mathbf{p}_a = \mathbf{f}_1 = \text{id} . \quad (14)$$

Proof. Eq. (14) follows immediately from the unitarity of S . For completeness, let us briefly argue that $\mathbf{p}_a \mathbf{p}_b = \delta_{a,b} \mathbf{p}_a$. We have

$$\begin{aligned} \mathbf{p}_a \mathbf{p}_b &= S_{1a} S_{1b} \sum_{g,h} \bar{S}_{ga} \bar{S}_{hb} \mathbf{f}_g \mathbf{f}_h \\ &= S_{1a} S_{1b} \sum_{g,h,j} \bar{S}_{ga} \bar{S}_{hb} N_{gh}^j \mathbf{f}_j \\ &= S_{1a} S_{1b} \sum_{g,h,j,x} \bar{S}_{ga} \bar{S}_{hb} \frac{S_{gx} S_{hx} S_{jx}}{S_{1x}} \mathbf{f}_j \end{aligned}$$

where we used the Verlinde formula (11) in the second step. With the unitarity of the S -matrix, we then obtain

$$\begin{aligned} \mathbf{p}_a \mathbf{p}_b &= S_{1a} S_{1b} \sum_{j,x} \delta_{a,x} \delta_{b,x} \frac{S_{jx}}{S_{1x}} \mathbf{f}_j \\ &= \delta_{a,b} S_{1a}^2 \sum_j \frac{S_{ja}}{S_{1a}} \mathbf{f}_j \\ &= \delta_{a,b} S_{1a} \sum_j S_{ja} \mathbf{f}_j . \end{aligned}$$

The claim follows from the symmetry property $S_{ja} = \bar{S}_{ja}$, see e.g., [24, Eq. (224)]. \square

As explained in the next section, TQFTs give rise to a representation of the Verlinde algebra. While the projections (introduced in Eq. (16) below) associated with the idempotents are not a basis for the logical algebra, they are a basis of a subalgebra isomorphic to the Verlinde algebra. This algebra must be respected by the locality-preserving unitaries, and this is best understood in terms of the idempotents. This is the origin of the non-trivial constraints we obtain on the realizable logical operators.

3.2 String-like operators and relations

Fix a simple closed curve $C : [0, 1] \rightarrow \Sigma$ on the surface. For each anyon label $a \in \mathbb{A}$ there is a “string-operator” $F_a(C)$ acting on $\mathcal{H}_{\text{phys}, \Sigma}$, supported in a constant-diameter neighborhood of C . In contrast to the abelian case, this will not generally be unitary. It corresponds to the process of creating a particle-antiparticle-pair (a, \bar{a}) , moving a along C , and subsequently

fusing to the vacuum. The last step in this process involves projection onto the ground space, which is not trivial in general: the operator $F_a(C)$ hence involves post-selection.

The operators $\{F_a(C)\}_{a \in \mathbb{A}}$ form a closed subalgebra $\mathcal{A}(C) \subset \mathcal{A}_\Sigma$: they preserve the ground space and satisfy

$$F_a(C)F_b(C) = \sum_n \delta_{abn} F_n(C) \quad \text{and} \quad F_a(C)^\dagger = F_{\bar{a}}(C) \quad \text{and} \quad F_1(C) = \text{id}_{\mathcal{H}_{\text{phys}}} . \quad (15)$$

A proof of (15) is straightforward in the diagrammatic formalism (but this is not needed here; we will use it as an axiom).

One important property we need is the following completeness relation:

Proposition 3.2 (Completeness of closed strings). *Consider an operator $O \in \mathcal{A}_\Sigma$ whose support is contained within a constant-diameter neighborhood of a simple loop C . Then $[O] = [\tilde{X}]$ for some $\tilde{X} \in \mathcal{A}(C)$. In other words, the logical action of O is identical to that of a linear combination of string-operators $F_a(C)$.*

This proposition can be seen as a consequence of the completeness condition for strings 2.2, the string deformation postulate 2.3 and the Verlinde algebra equation (15). A similar argument leads us to the following conclusion in the scenario where strings may not end at boundaries.

Proposition 3.3 (Completeness of homology classes). *The full logical algebra $[\mathcal{A}_\Sigma]$ is generated by the logical algebras $[\mathcal{A}(C)]$ associated with a finite number of inequivalent non-contractible simple² loops C .*

Proof. That the algebra $[\mathcal{A}_\Sigma]$ is finite-dimensional can be seen from the finite dimensionality of the code space \mathcal{H}_Σ . By Postulate 2.2, we know that the algebra $[\mathcal{A}_\Sigma]$ is generated by $\{\mathcal{A}(C)\}_C$. Let us start from a trivial algebra and build up $[\mathcal{A}_\Sigma]$ from a finite number of loops. As long as the algebra is not complete, we may include additional loops C such that $[\mathcal{A}(C)]$ is not included in the partially generated algebra. Such a loop C must be inequivalent to the previously included loops due to Postulate 2.3. After a number of steps no greater than the square of the ground space dimension, we will have constructed the complete algebra. \square

Eq. (15) shows that the collection of operators $\{F_a(C)\}_{a \in \mathbb{A}}$ form a representation of the Verlinde (fusion) algebra Ver . This will be central in the following development. Considering the primitive idempotents (12), it is natural to consider the corresponding operators in this representation, that is, we set

$$P_a(C) = S_{1a} \sum_b \bar{S}_{ba} F_b(C) . \quad (16)$$

Since the set $\{F_a(C)\}_{a \in \mathbb{A}}$ forms a representation of the Verlinde algebra, the $\{P_a(C)\}_{a \in \mathbb{A}}$ are orthogonal projectors as shown in Proposition 3.1. The inverse relation to (16) is given by

$$F_b(C) = \sum_a \frac{S_{ba}}{S_{1a}} P_a(C) . \quad (17)$$

²Note that on the punctured sphere, simple loops are characterized by the set of punctures they contain in their interior.

While the projectors $P_a(C)$ associated with a loop do not span the full logical algebra, they do span the fusion algebra, locally observable on C , of the underlying TQFT which must be respected by locality preserving unitaries. Intuitively, $\{P_a(C)\}_{a \in \mathbb{A}}$ are projectors onto the smallest possible sectors of the Hilbert space which can be distinguished by a measurement supported on C . This is the origin of the non-trivial constraints we obtain on the realizable logical operators.

3.3 Bases of the Hilbert space \mathcal{H}_Σ

A state in the image of $P_a(C)$ has the interpretation of carrying flux a through the loop C . In particular, since the code space \mathcal{H}_Σ corresponds to the vacua of a TQFT, there are no anyons present on Σ , however, there can be flux associated to non-contractible loops. We can use the operators $\{P_a(C)\}_{a,C}$ to define bases of the Hilbert space \mathcal{H}_Σ .

Let us first define the Hilbert space \mathcal{H}_Σ in more detail.

Definition 3.1 (DAP-decomposition). *Consider a maximal collection $\mathcal{C} = \{C_j \mid C_j : [0, 1] \rightarrow \Sigma\}_j$ of pairwise non-intersecting non-contractible loops, which cut the surface Σ into a collection of surfaces homeomorphic to discs, annuli and pants. We call \mathcal{C} a DAP-decomposition.*

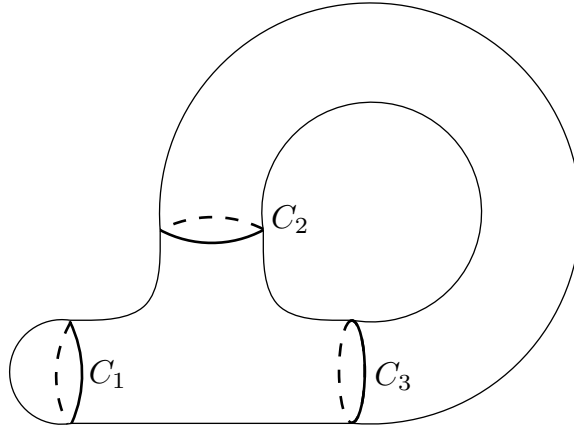


Figure 4: A simple DAP decomposition of a torus utilizing a disc enclosed by C_1 , an annulus enclosed by $\{C_2, C_3\}$ and a pair of pants enclosed by $\{C_1, C_2, C_3\}$. This decomposition is not minimal in that the same manifold could have been decomposed using a single loop.

A labeling $\ell : \mathcal{C} \mapsto \mathbb{A}$ is an assignment of an anyon label $\ell(C)$ to every loop $C \in \mathcal{C}$ of a DAP decomposition. We call ℓ fusion-consistent if it satisfies the following conditions:

- (i) for every loop $C \in \mathcal{C}$ enclosing a disc on Σ , $\ell(C) = 1$, the vacuum label of the anyon model.
- (ii) for every pair of loops $\{C_2, C_3\} \subset \mathcal{C}$ defining an annulus in Σ , $\ell(C_2) = \bar{\ell}(C_3)$ assuming the loops are oriented such that the annulus is found to the left.
- (iii) for every triple $\{C_1, C_2, C_3\} \subset \mathcal{C}$ defining a pair of pants in Σ , the labels ℓ satisfy the fusion rule $N_{\ell(C_1), \ell(C_2)}^{\ell(C_3)} \neq 0$, where the loops are oriented such that the pair of pants is found to the left.

Here we may assume $\ell(\bar{C}) = \bar{\ell}(C)$, where \bar{C} denotes the loop coinciding with C but with opposite orientation.

Now fix any DAP-decomposition \mathcal{C} of Σ and let $\mathbf{L}(\mathcal{C}) \subset \mathbb{A}^{\mathcal{C}}$ be the set of fusion-consistent labelings. The Hilbert space \mathcal{H}_{Σ} is the formal span of elements of $\mathbf{L}(\mathcal{C})$. That is, it consists of complex-valued functions

$$\mathcal{H}_{\Sigma} := \{ \Psi \mid \Psi : \mathbf{L}(\mathcal{C}) \rightarrow \mathbb{C} \} .$$

Any fusion-consistent labeling $\ell \in \mathbf{L}(\mathcal{C})$ defines an element $\Psi_{\ell} \in \mathcal{H}_{\Sigma}$ by $\Psi_{\ell}(\ell') = \delta_{\ell, \ell'}$. The vectors $\{\Psi_{\ell}\}_{\ell \in \mathbf{L}(\mathcal{C})}$ are an orthonormal basis (which we call $\mathcal{B}_{\mathcal{C}}$) of \mathcal{H}_{Σ} , and this defines the inner product.

It is important to remark that this construction of \mathcal{H}_{Σ} is independent of the DAP-decomposition \mathcal{C} of Σ in the following sense: if \mathcal{C} and \mathcal{C}' are two distinct DAP-decompositions, then there is a unique unitary basis change between the bases $\mathcal{B}_{\mathcal{C}}$ and $\mathcal{B}_{\mathcal{C}'}$. The basis change can be obtained as a product of unitaries associated with local “moves” connecting two DAP decompositions \mathcal{C} and \mathcal{C}' . One such basis change is associated with a four-punctured sphere (the F -move), and specified by the unitary F -matrix in Fig. 5. The second matrix of this kind, the S -matrix, connects the two bases $\mathcal{B}_{\{C_1\}}$ and $\mathcal{B}_{\{C_2\}}$ of $\mathcal{H}_{\text{torus}}$ associated with the first and second non-trivial cycles on the torus Fig. 5. In this case, writing $|a\rangle_j \in \mathcal{B}_{\{C_j\}}$ for $j = 1, 2$ since each basis element Ψ_{ℓ} is specified by a single label $\ell(C_j) = a \in \mathbb{A}$, we have the relation

$$|a\rangle_2 = \sum_b S_{b,a} |b\rangle_1 . \quad (18)$$

A basis element $\Psi_{\ell} \in \mathcal{B}_{\mathcal{C}}$ associates the anyon label $\ell(C)$ with each curve $C \in \mathcal{C}$. The vector Ψ_{ℓ} spans the simultaneous $+1$ -eigenspace of the projections $\{P_{\ell(C)}\}_{C \in \mathcal{C}}$. It is also a simultaneous eigenvector with respect to Dehn-twists along each curve $C \in \mathcal{C}$ with eigenvalue $e^{i\theta_{\ell(C)}}$. The action of Dehn-twists along curves C' not belonging to \mathcal{C} can be obtained by applying the local moves to change into a basis $\mathcal{B}_{\mathcal{C}'}$ associated with a DAP-decomposition \mathcal{C}' containing C' .

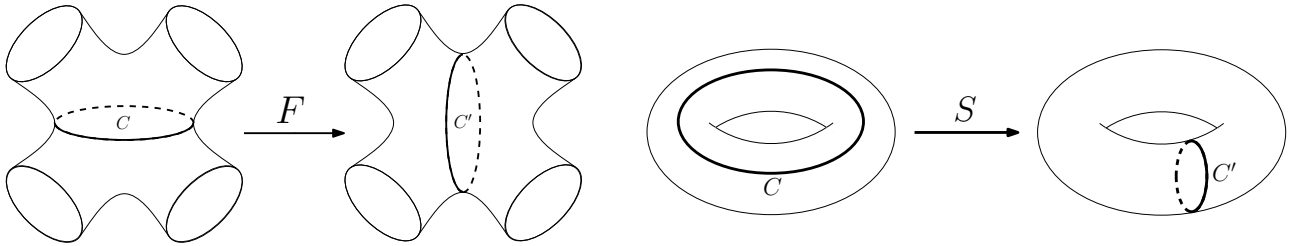


Figure 5: Two DAP-decompositions $\mathcal{C} = \{C\}$ and $\mathcal{C}' = \{C'\}$ of either the 4-punctured sphere (left), or the torus (right), are related by an F -move or an S -move, respectively.

3.4 Open surfaces: labeled boundaries

So far, we have been discussing the Hilbert space \mathcal{H}_{Σ} associated with closed surfaces; this does not cover the physically important case of pinned localized excitations (which correspond to

punctures/holes in the surface). Here we describe the modifications necessary to deal with surfaces with boundaries. We assume that the boundary $\partial\Sigma = \bigcup_{\alpha=1}^M \hat{C}_\alpha$ is the disjoint union of M simple closed curves, and assume that an orientation $\hat{C}_\alpha : [0, 1] \rightarrow \partial\Sigma$ has been chosen for each boundary component \hat{C}_α such that Σ is found to the left. In addition, we fix a label $a_\alpha \in \mathbb{A}$ for every boundary component \hat{C}_α . We call this a labeling of the boundary. Let us write $\Sigma(a_1, \dots, a_M)$ for the resulting object (i.e., the surfaces, its oriented boundary components, and the associated labels). We call $\Sigma(a_1, \dots, a_M)$ a surface with labeled boundary components; slightly abusing notation, we sometimes write $\Sigma = \Sigma(a_1, \dots, a_M)$ when the presence of boundaries is understood/immaterial.

A TQFT associates to every surface $\Sigma(a_1, \dots, a_M)$ with labeled boundary components a Hilbert space $\mathcal{H}_{\Sigma(a_1, \dots, a_M)}$. The construction is analogous to the case of closed surfaces and based on DAP-decompositions. The only modification compared to the case of closed surfaces is that only DAP-decompositions including the curves $\{\hat{C}_\alpha\}_{\alpha=1}^M$ are allowed; furthermore, the labeling on these boundary components is fixed by $\{a_\alpha\}_{\alpha=1}^M$. That is, “valid” DAP-decompositions are of the form $\mathcal{C} = \{C_1, \dots, C_N, \hat{C}_1, \dots, \hat{C}_M\}$ with curves $\{C_j\}_{j=1}^N$ “complementing” the boundary components, and valid labelings are fusion-consistent, i.e., $\ell \in \mathcal{L}(C)$ with the additional condition that they agree with the boundary labels, $\ell(\hat{C}_\alpha) = a_\alpha$ for $\alpha = 1, \dots, M$. To simplify the discussion, we will often omit the boundary components $\{\hat{C}_\alpha\}_\alpha$ and focus on the remaining degrees of freedom associated with the curves $\{C_j\}_j$. It is understood that boundary labelings have to be fusion-consistent with the labeling $\{a_\alpha\}_\alpha$ of the boundary under consideration.

As a final remark, note that boundary components labeled with the trivial particle $1 \in \mathbb{A}$ correspond to contractible loops in a surface without this boundary (i.e., obtained by “gluing in a disc”). This means, that they can be omitted: we have the isomorphism

$$\mathcal{H}_{\Sigma(1)} \cong \mathcal{H}_{\Sigma'} ,$$

where Σ' is the surface with one boundary component less than that of Σ .

Example: the M -anyon Hilbert space

A typical example we are interested in is the labeled surface

$$S^2(z^M) = S^2(\underbrace{z, \dots, z}_{M \text{ times}}) ,$$

where $S^2(, , \dots , ,)$ is the punctured sphere, and $z \in \mathbb{A}$ is some fixed anyon type (we assume that each boundary component has the same orientation). The Hilbert space $\mathcal{H}_{S^2(z^M)}$ is the space of M anyons of type z . When $M = N + 3$ for some $N \in \mathbb{N}$, we can choose a ‘standard’ DAP-decomposition $\mathcal{C} = \{C_j\}_{j=1}^N$ as shown in Fig. 6. A fusion-consistent labeling ℓ of the standard DAP-decomposition \mathcal{C} corresponds to a sequence $(x_1, \dots, x_N) = (\ell(C_1), \dots, \ell(C_N))$ such that

$$\delta_{zzx_1} = \delta_{x_N z \bar{z}} = 1 \quad \text{and} \quad \delta_{x_j z x_{j+1}} = 1 \quad \text{for all } j = 1, \dots, N-1, \quad (19)$$

as illustrated by figure 6.

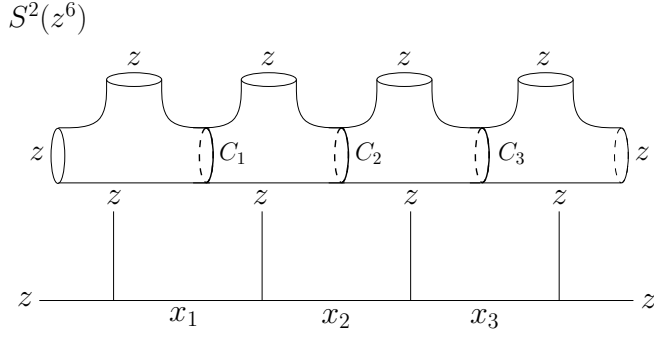


Figure 6: The ‘standard’ DAP-decomposition of the 6-punctured sphere, and the corresponding fusion-tree notation representing the labeling which assigns $\ell(C_i) = x_i$.

3.5 The gluing axiom

Consider a closed curve C embedded in Σ . We will assume that C is an element of a DAP-decomposition \mathcal{C} ; although this is not strictly necessary, it will simplify our discussion. Now consider the surface Σ' obtained by cutting Σ along C . Compared to Σ , this is a surface with two boundary components C'_1, C'_2 (both isotopic to C) added. We will assume that these have opposite orientation. A familiar example is the case where cutting Σ along C results in two disconnected surfaces $\Sigma' = \Sigma_1 \cup \Sigma_2$, as depicted in Fig. 7 in the case where Σ is the 4-punctured sphere.

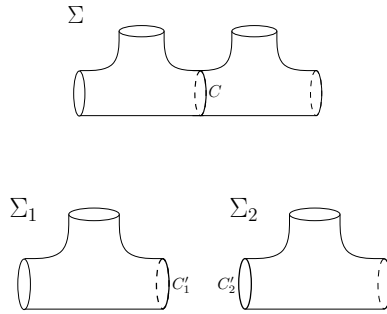


Figure 7: Cutting a surface Σ along some closed curve C of a DAP-decomposition yields a disconnected surface $\Sigma' = \Sigma_1 \cup \Sigma_2$ having additional boundary components C'_1 and C'_2

Let a be a particle label. We will denote by $\mathcal{H}_{\Sigma'(a,\bar{a})}$ the Hilbert space associated with the open surface Σ' , where boundary C'_1 is labeled by a and boundary C'_2 by \bar{a} . The gluing axiom states that the Hilbert space of the surface Σ has the form

$$\mathcal{H}_{\Sigma} \cong \bigoplus_a \mathcal{H}_{\Sigma'(a,\bar{a})} \quad (20)$$

where the direct sum is over all particle labels a that occur in different fusion-consistent labelings of \mathcal{C} . In the special case where cutting along C gives two components Σ_1, Σ_2 , we have $\mathcal{H}_{\Sigma} \cong \bigoplus_a \mathcal{H}_{\Sigma_1(a)} \otimes \mathcal{H}_{\Sigma_2(\bar{a})}$.

The isomorphism (20) can easily be made explicit. A first observation is that \mathcal{H}_Σ decomposes as $\mathcal{H}_\Sigma = \bigoplus_a \mathcal{H}_{a,\Sigma}(C)$, where

$$\mathcal{H}_{a,\Sigma}(C) := \text{span}\{\Psi_\ell \mid \ell \in \mathbf{L}(\mathcal{C}), \ell(C) = a\} \quad (21)$$

is the space spanned by all labelings which assign the label a to C . It therefore suffices to argue that

$$\mathcal{H}_{a,\Sigma}(C) \cong \mathcal{H}_{\Sigma'(a,\bar{a})} . \quad (22)$$

To do so, observe that the DAP-decomposition \mathcal{C} of Σ gives rise to a DAP-decomposition $\mathcal{C}' = \mathcal{C} \setminus \{C\}$ of Σ' . Any labeling $\ell \in \mathbf{L}(\mathcal{C})$ with $\ell(C) = a$ restricts to a labeling $\ell' \in \mathbf{L}(\mathcal{C}')$ of the labeled surface $\Sigma'(a, \bar{a})$. Conversely, any labeling $\ell' \in \mathbf{L}(\mathcal{C}')$ of the surface $\Sigma'(a, \bar{a})$ provides a labeling $\ell \in \mathbf{L}(\mathcal{C})$ (by setting $\ell(C) = a$). This defines the isomorphism (22) in terms of basis states $\{\Psi_\ell\}_{\ell \in \mathbf{L}(\mathcal{C})}$ and $\{\Psi_{\ell'}\}_{\ell' \in \mathbf{L}(\mathcal{C}')}$.

Example: decomposing the M -anyon Hilbert space

Consider the M -punctured sphere $\Sigma = S^2(z^M)$ with the standard DAP decomposition of Fig. 6 and boundary labels z (corresponding to M anyons of type z). Cutting $S^2(z^M)$ along C_j gives a surface Σ'_j which is the disjoint union of two punctured spheres, with $j+2$ and $M-j$ punctures, respectively. The resulting surface labelings are $S^2(z^{j+1}, a)$ and $S^2(\bar{a}, z^{M-1-j})$. That is, if $\Sigma = S^2(z^M)$ is the original surface and $\Sigma'_j(a, \bar{a})$ is the resulting one, then

$$\mathcal{H}_{\Sigma'_j(a,\bar{a})} = \mathcal{H}_{S^2(z^{j+1},a)} \otimes \mathcal{H}_{S^2(\bar{a},z^{M-1-j})} . \quad (23)$$

This is illustrated in Fig. 8 for the case $M = 6$ and $j = 2$.

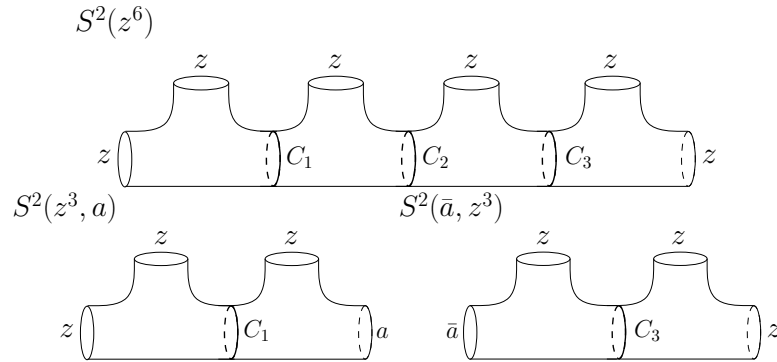


Figure 8: The 6-punctured sphere $S^2(z^6)$ shown with three curves $C_1, C_2, C_3 \in \mathcal{C}$ of a DAP-decomposition. Cutting along C_2 with labeling $\ell(C_2) = a$ results in the two surfaces $S^2(z^3, a)$ and $S^2(\bar{a}, z^3)$.

4 Locality-preserving automorphisms for TQFTs

In this section, we derive restrictions on topologically protected gates for general non-abelian models. Our strategy will again be to consider what happens to string-operators. It is tempting to try to apply similar arguments as in the proof of Theorem 2.11 to characterize protected gates for non-abelian models. Unfortunately, this proof does not directly generalize since it relies on properties of string-operators that are special to abelian anyons. For example, the string-operators are non-unitary, and, in addition, the decoupling Postulate 2.4 will not apply in general non-abelian models. In particular, it is no longer true that an operator of the form $F_a(C)F_b(C')F_{\bar{a}}(C)F_{\bar{b}}(C')$ (for two conjugate loops C and C') is “cleanable”, i.e., has the same logical action as a local operator.

We will first consider operators associated with a single loop C , and derive restrictions on the ‘fattening’ map $F_a(C) \mapsto \mathbf{F}_a(C) := UF_a(C)U^\dagger$, or, more precisely, its effect on logical operators, $[F_a(C)] \mapsto [\mathbf{F}_a(C)] = [UF_a(C)U^\dagger]$. We will argue that this map implements an isomorphism of the Verlinde algebra and exploit this fact to derive a constraint which is ‘local’ to a specific loop. We will subsequently consider more ‘global’ constraints arising from fusion rules, as well as basis changes.

4.1 Locality-preserving unitaries and symmetries of anyon model

We would like to characterize locality-preserving unitary automorphisms $U \in \mathcal{A}_\Sigma$ in terms of their logical action $[U]$. A first goal is to characterize the map

$$\begin{aligned} \rho_U : \quad [\mathcal{A}_\Sigma] &\rightarrow [\mathcal{A}_\Sigma] \\ [X] &\mapsto [UXU^{-1}] , \end{aligned} \quad (24)$$

which determines the evolution of logical observables in the Heisenberg picture. (Clearly, this does not depend on the representative, i.e., if $[X] = [X']$, then $\rho_U([X]) = \rho_U([X'])$. In fact, the map (24) fully determines U up to a global phase since $[\mathcal{A}_\Sigma]$ contains an operator basis for linear maps on \mathcal{H}_Σ . However, it will often be more informative to characterize the action of $[U]$ on basis elements of \mathcal{H}_Σ . This will require additional effort.

The main observation is that the map (24) defines an automorphism of $[\mathcal{A}_\Sigma]$, since,

$$\rho_U([X])\rho_U([X']) = \rho_U([X][X']) \quad \text{for all } X, X' \in \mathcal{A}_\Sigma \quad \text{and} \quad \rho_U^{-1} = \rho_{U^{-1}} . \quad (25)$$

Combined with the locality of U , (25) severely constrains ρ_U .

4.2 A local constraint from a simple closed loop

Specifying the action of ρ_U on all of $[\mathcal{A}_\Sigma]$ completely determines $[U]$ up to a global phase. However, this is not entirely straightforward; instead, we fix some simple closed curve C and characterize the restriction to the subalgebra $\mathcal{A}(C) \subset \mathcal{A}_\Sigma$, i.e., the map

$$\begin{aligned} \rho_U(C) : \quad [\mathcal{A}(C)] &\rightarrow [\mathcal{A}(C)] \\ [X] &\mapsto [UXU^{-1}] , \end{aligned} \quad (26)$$

Observe that this map is well-defined since UXU^{-1} is supported in a neighborhood of C (by the locality-preservation of U), and hence $[UXU^{-1}] = [X']$ for some operator $X' \in \mathcal{A}(C)$ (here

we have used Proposition 3.2). It is also easy to see that it defines an automorphism of the subalgebra $[\mathcal{A}(C)]$.

As we argued above, the algebra $\mathcal{A}(C)$ decomposes into $|\mathbb{A}|$ copies of \mathbb{C} with idempotents $\{P_a(C)\}_{a \in \mathbb{A}}$. This carries over to $[\mathcal{A}(C)] \cong \mathbb{C}^{\oplus |\mathbb{A}|}$, which has idempotents $\{[P_a(C)]\}_{a \in \mathbb{A}}$. We use the following fact:

Lemma 4.1. *The set of automorphisms of the algebra $\mathbb{C}^{\oplus N}$ is in one-to-one correspondence with the permutations S_N . For $\pi \in S_N$, the associated automorphism $\rho_\pi : \mathbb{C}^{\oplus N} \rightarrow \mathbb{C}^{\oplus N}$ is defined by its action*

$$\rho_\pi(\mathbf{p}_j) = \mathbf{p}_{\pi(j)} \quad \text{for } j = 1, \dots, N \quad (27)$$

on the central idempotents $\mathbf{p}_j := 0^{\oplus j-1} \oplus 1 \oplus 0^{\oplus (N-j-1)}$.

Proof. It is clear that (27) defines an automorphism for every $\pi \in S_N$. Conversely, suppose that $\rho(\mathbf{p}_j) = \sum_k \rho_{k,j} \mathbf{p}_k$ is an automorphism. Then we must have $\rho(\mathbf{p}_j)^2 = \rho(\mathbf{p}_j)$ which implies that $\rho_{k,j}^2 = \rho_{k,j}$ or $\rho_{k,j} \in \{0, 1\}$ for all j, k . The orthogonality $\rho(p_j)\rho(p_k) = 0$ for $j \neq k$ implies that the columns of the matrix $(\rho_{k,j})_{k,j}$ must be orthogonal (with respect to the Euclidean inner product). Since each column consists of 0s and 1s only, and ρ is an isomorphism, this is only possible if each column contains exactly one 1. \square

Applying this to $[\mathcal{A}(C)]$ gives the following result. It shows that a locality-preserving automorphism realizes, up to important phases, a symmetry of the underlying fusion category (i.e., it permutes fusion-consistent labelings). Let us emphasize that it is the projectors (idempotents) $P_a(C)$ which are being permuted, and not the string operators $F_a(C)$.

Proposition 4.1 (Local constraint). *Let U be a locality-preserving automorphism of the code, and let $\rho_U([X]) = [UXU^{-1}]$.*

(i) *For each simple closed loop C on Σ , there is a permutation $\pi^C : \mathbb{A} \rightarrow \mathbb{A}$ of the particle labels such that*

$$\begin{aligned} \rho_U : [\mathcal{A}(C)] &\rightarrow [\mathcal{A}(C)] \\ [P_a(C)] &\mapsto [P_{\pi^C(a)}(C)] \end{aligned} \quad \text{for all } a \in \mathbb{A} , \quad (28)$$

(and linearly extended to all of $[\mathcal{A}(C)]$).=

(ii) *For some anyon model \mathbb{A} with an associated S matrix, let $D_{a,b} = \delta_{a,b} \cdot d_a$ be the diagonal matrix with the quantum dimensions on the diagonal. Let $\pi = \pi^C : \mathbb{A} \rightarrow \mathbb{A}$ be a permutation associated with a loop C as in (i), and let Π be the matrix defined by $\Pi_{x,y} := \delta_{x,\pi(y)}$. Define the matrix*

$$\Lambda := S\Pi^{-1}D\Pi D^{-1}\Pi^{-1}S^{-1} . \quad (29)$$

Then

$$\rho_U([F_b]) = \sum_{b'} \Lambda_{b,b'} [F_{b'}] . \quad (30)$$

Proof. We have already argued that (i) holds. For the proof of (ii), we use the relationship between $\{P_a(C)\}_a$ and $\{F_a(C)\}_a$ (cf. (16) and (17)) to get (suppressing the dependence on the loop C)

$$\rho_U([F_b]) = \sum_a \frac{S_{b,a}}{S_{1,a}} [P_{\pi(a)}] = \sum_{b'} \left(\sum_a \frac{S_{b,a}}{S_{1,a}} S_{1,\pi(a)} \bar{S}_{b',\pi(a)} \right) [F_{b'}] .$$

The claim (30) follows from this using $(\Pi^{-1}S^{-1})_{a,b'} = (S^{-1})_{\pi(a),b'} = \bar{S}_{b',\pi(a)}$ by the unitarity of S , as well as the fact that $S_{1,a} = d_a/\mathcal{D}$ and hence $\frac{S_{b,a}}{S_{1,a}} S_{1,\pi(a)} = (S\Pi^{-1}D\Pi D^{-1})_{b,a}$. \square

From this proposition alone, we may already distill an Eastin and Knill [12] type statement, which is one of our main conclusions.

Corollary 4.2 (Finite-group of protected gates). *The set of locality-preserving automorphisms for the code \mathcal{H}_Σ defined by a TQFT on an orientable manifold Σ generates a finite group (up to irrelevant global phases).*

In particular, this means that locality-preserving automorphisms on their own do not provide quantum computational universality.

Proof. The proof hinges on Proposition 3.3 that the full algebra of logical operators \mathcal{A}_Σ is generated by the logical algebras $\mathcal{A}(C)$ associated to a finite³ number n of loops C . Within each loop, the action of a locality-preserving unitary is specified by the permutation it performs on the corresponding idempotents labeled by the anyon labels \mathbb{A} . The composition of two such unitaries is specified by the composition of these permutations, and hence, the set of logical unitaries generated is bounded by $(|\mathbb{A}|!)^n$, which is finite. \square

4.3 A global constraint: DAP-decompositions, fusion rules and the gluing axiom

For higher-genus surfaces, we can obtain information by applying Proposition 4.1 to all loops of a DAP-decomposition; these must then satisfy the following consistency condition.

Proposition 4.2 (Global constraint from fusion rules). *Let U be a locality-preserving automorphism of the code. Let \mathcal{C} be a DAP-decomposition of Σ , and consider the family of permutations $\vec{\pi} = \{\pi^C\}_{C \in \mathcal{C}}$ defined by Proposition 4.1. Then this defines a permutation*

$$\begin{aligned} \vec{\pi} : \mathbf{L}(\mathcal{C}) &\rightarrow \mathbf{L}(\mathcal{C}) \\ \ell &\mapsto \vec{\pi}(\ell)(C) := \pi^C(\ell(C)) \quad \text{for all } C \in \mathcal{C} \end{aligned} \tag{31}$$

of the set $\mathbf{L}(\mathcal{C})$ of fusion-consistent labelings. We have

$$U\Psi_\ell = e^{i\varphi(\ell)} \Psi_{\vec{\pi}(\ell)} \quad \text{for all } \ell \in \mathbf{L}(\mathcal{C}) \tag{32}$$

with some phase $e^{i\varphi(\ell)}$ depending on ℓ .

³We expect that the number of independent generators of the mapping class group, which for closed manifolds are no more than $2g + 1$ [20, 30], are sufficient, but this is not essential to our statement.

Proof. Let us fix some basis element $\Psi_\ell \in \mathcal{B}_\mathcal{C}$. The vector Ψ_ℓ is a +1-eigenvector of $P_{\ell(C)}(C)$ for each $C \in \mathcal{C}$; hence according to (28), the vector $U\Psi_\ell$ is a +1-eigenvector of $P_{\pi^C(\ell(C))}(C) = P_{\vec{\pi}(\ell)(C)}(C)$ for every $C \in \mathcal{C}$. This implies that it is proportional to $\Psi_{\vec{\pi}(\ell)}$, hence we obtain (32). Fusion-consistency of $\vec{\pi}(\ell)$ follows because $U\Psi_\ell$ must be an element of \mathcal{H}_Σ . \square

Proposition (4.2) expresses the requirement that a locality-preserving automorphism U maps the set of fusion-consistent labelings into itself.

In fact, we can say more: it must be an isomorphism between the subspaces of \mathcal{H}_Σ arising from the gluing axiom (i.e., Eq. (20)). This allows us to constrain the set of allowed permutations $\pi = \{\pi^C\}_{C \in \mathcal{C}}$ arising from locality-preserving automorphisms even further:

Proposition 4.3 (Global constraint from gluing). *Let C be an element of a DAP-decomposition of Σ . Recall that*

$$\mathcal{H}_\Sigma = \bigoplus_a \mathcal{H}_{a,\Sigma}(C) , \quad (33)$$

where the subspaces in the direct sum are defined by labelings associating a to C . Let U be a locality-preserving automorphism of the code and let $\pi^C : \mathbb{A} \rightarrow \mathbb{A}$ be the permutation associated with C by Proposition 4.1. Then for every $a \in \mathbb{A}$ occuring Eq. (33), the restriction of U to $\mathcal{H}_{a,\Sigma}(C)$ defines an isomorphism

$$\mathcal{H}_{a,\Sigma}(C) \cong \mathcal{H}_{\pi^C(a),\Sigma}(C) . \quad (34)$$

In particular, if Σ' is the surface obtained by cutting Σ along C , then

$$\mathcal{H}_{\Sigma'(a,\bar{a})} \cong \mathcal{H}_{\Sigma'(\pi^C(a),\overline{\pi^C(a)})} \quad (35)$$

for every $a \in \mathbb{A}$ occuring in the sum (33).

The reason we refer to Proposition (4.3) as a global constraint (even though it superficially only concerns a single curve C) is that the surface Σ' and hence the spaces (35) depend on the global form of the surface Σ outside the support of C .

Proof. Proposition (4.2) implies that $U\mathcal{H}_{a,\Sigma}(C) \subset \mathcal{H}_{\pi^C(a),\Sigma}(C)$ for any a in expression (33). Since U acts unitarily on the whole space \mathcal{H}_Σ , this is compatible with (33) only if $U\mathcal{H}_{a,\Sigma}(C) = \mathcal{H}_{\pi^C(a),\Sigma}(C)$ for any such a . This proves (34). Statement (35) then immediately follows from (22). \square

A simple but useful implication of Proposition 4.3 is that

$$\dim(\mathcal{H}_{\Sigma'(a,\bar{a})}) = \dim\left(\mathcal{H}_{\Sigma'(\pi^C(a),\overline{\pi^C(a)})}\right) \quad (36)$$

is a necessary condition that π^C has to satisfy.

4.4 Additional global constraints: DAP-decompositions and basis changes

Eq. (28) essentially tells us that a locality-preserving protected gate U can only permute particle labels; it indicates that such a gate U is related to symmetries of the anyon model. But (28) does not tell us what phases basis states may acquire. In this section, we show how to obtain constraints on these phases by considering basis changes. This also further constrains the allowed permutations on the labels of the idempotents.

Consider two DAP-decompositions \mathcal{C} and \mathcal{C}' . Expressed in the first basis $\mathcal{B}_{\mathcal{C}}$, we have

$$U\Psi_{\ell} = e^{i\varphi_{\ell}}\Psi_{\pi(\ell)} \quad (37)$$

for some unknown phase $\varphi_{\ell} \in \mathbb{C}$ depending only on the labeling $\ell \in \mathcal{L}(\mathcal{C})$. This means that with respect to the basis elements of $\mathcal{B}_{\mathcal{C}}$, the operator U is described by a matrix $\mathbf{U} = \mathbf{\Pi}\mathbf{D}(\{\varphi_{\ell}\}_{\ell})$, where $\mathbf{\Pi}$ is a permutation matrix (acting on the fusion-consistent labelings $\mathcal{L}(\mathcal{C})$), and \mathbf{D} is a diagonal matrix with entries $\{e^{i\varphi_{\ell}}\}_{\ell}$ on the diagonal.

Analogously, we can consider the operator U expressed as a matrix \mathbf{U}' in terms of the basis elements of $\mathcal{B}_{\mathcal{C}'}$. We conclude that $\mathbf{U}' = \mathbf{\Pi}'\mathbf{D}(\{\varphi'_{\ell}\}_{\ell})$, for $\ell \in \mathcal{L}(\mathcal{C}')$, with a (potentially different) permutation matrix $\mathbf{\Pi}'$, and (potentially different) phases $\{\varphi'_{\ell}\}_{\ell}$.

Let \mathbf{V} be the unitary change-of-basis matrix (obtained e.g., from a sequence of F -moves or the S -matrix) for going from $\mathcal{B}_{\mathcal{C}}$ to $\mathcal{B}_{\mathcal{C}'}$. Then we must have

$$\mathbf{U}'\mathbf{V} = \mathbf{V}\mathbf{U} . \quad (38)$$

We show below that Eq. (38) strongly constrains the phases in (32).

For concreteness and simplicity, we consider the torus and the four-punctured sphere: for each of these surfaces, there are only two inequivalent DAP-decompositions (and hence there is only one basis change \mathbf{V} to consider). More generally (e.g., for the 5-punctured sphere), we need to consider several different basis changes and obtain a constraint of the form (38) for every pair of bases. This is described in Section 4.4.4.

4.4.1 Determining phases for the torus

For the torus, where each DAP decomposition $\mathcal{C} = \{C_j\}$ consists of a single cycle (either $j = 1$ or $j = 2$), the fusion-consistent labelings are simply the set of anyon labels, $\mathcal{L}(\mathcal{C}) = \mathbb{A}$. Hence \mathbf{U} and \mathbf{U}' are $|\mathbb{A}| \times |\mathbb{A}|$ -matrices. In particular, the analog to (37) is

$$U\Psi_a = e^{i\varphi_a}\Psi_{\pi(a)} \quad (39)$$

for some permutation $\pi : \mathbb{A} \rightarrow \mathbb{A}$ and phases $\{\varphi_a\}_{a \in \mathbb{A}}$. The basis change \mathbf{V} is the S -matrix. The consistency equation (38) becomes

$$\mathbf{\Pi}'\mathbf{D}(\{\varphi'_{\ell}\}_{\ell})S = S\mathbf{\Pi}\mathbf{D}(\{\varphi_{\ell}\}_{\ell}) \quad (40)$$

We are looking for permutations $\mathbf{\Pi}, \mathbf{\Pi}' : \mathbb{A} \rightarrow \mathbb{A}$ and phases $\{\varphi_a\}_{a \in \mathbb{A}}, \{\varphi'_a\}_{a \in \mathbb{A}}$ satisfying this equation, as every locality-preserving automorphism U gives rise to such a solution.

4.4.2 Determining phases for the four-punctured sphere: fixed boundary labels

For a four-punctured sphere Σ , we can fix the labels on the punctures to $i, j, k, l \in \mathbb{A}$. The corresponding space $\mathcal{H}_{\Sigma(i,j,k,l)}$ associated to this open surface with labeled boundary components is the fusion space V_{kl}^{ij} . (In the non-abelian case, this space can have dimension larger than 1.) We have two bases $\mathcal{B}_C, \mathcal{B}_{C'}$ of this fusion space, corresponding to two different DAP-decompositions differing by one loop. We can enumerate basis elements by the label assigned to this loop. Let $\{\Psi_a\}_a$ and $\{\Psi'_a\}_a$ be the elements of the basis \mathcal{B}_C and $\mathcal{B}_{C'}$, respectively. Note that a ranges over all elements consistent with the fusion rules.

For the models without fusion degeneracy being considered, this is, $\delta_{ija} = \delta_{kla} = 1$. Let $Q = Q(i, j, k, l)$ be the set of such elements. The basis change is given by the F -matrix

$$\Psi'_m = \sum_n F_{kln}^{ijm} \Psi_n .$$

Considering a locality-preserving automorphism which preserves the boundary labels (this is reasonable if we think of them as certain boundary conditions of the system), we can apply the procedure explained above to find the action

$$U\Psi_a = e^{i\varphi_a} \Psi_{\pi(a)}$$

on basis states. Here $\pi : Q \rightarrow Q$ permutes fusion-consistent labels. To apply the reasoning above, we have to use the $|Q \times Q|$ -basis change matrix \mathbf{V} defined by $\mathbf{V}_{m,n} = F_{kln}^{ijm}$.

Solving the consistency relation (38) (for the permutations π, π' and phases $\{\varphi_a\}_a, \{\varphi'_a\}_a$) shows that for any permutation π that is part of a solution, the function φ_a takes the form

$$\varphi_a = \eta + f(a) , \tag{41}$$

where η is a global phase and f belongs to a certain set of functions which we denote

$$\text{Iso} \left(\underline{j \quad i \quad . \quad l \quad k} \rightarrow \underline{j \quad i \quad \pi(\cdot) \quad l \quad k} \right) . \tag{42}$$

(The reason for this notation will become clearer when we discuss isomorphisms in the next section; here we are concerned with relative phases arising from automorphisms.) In summary, we have

$$U|\Psi_a\rangle = e^{i\eta} e^{if(a)} |\Psi_{\pi(a)}\rangle \quad \text{where } f \in \text{Iso} \left(\underline{j \quad i \quad . \quad l \quad k} \rightarrow \underline{j \quad i \quad \pi(\cdot) \quad l \quad k} \right) . \tag{43}$$

Here the set (42) can be computed by solving the consistency relation

$$\mathbf{V}\Pi\mathbf{D}(\{\varphi_a\}_a) = \Pi'\mathbf{D}(\{\varphi'_a\}_a)\mathbf{V} \tag{44}$$

with $\mathbf{V}_{m,n} = F_{kln}^{ijm}$. This scenario is depicted as a special case of the commutative diagram displayed in Fig. 9.

4.4.3 Determining phases for the four-punctured sphere in general

Consider the four-punctured sphere Σ with fixed labels $i, j, k, l \in \mathbb{A}$ on the punctures. Let $\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}$ be another set of labels such that the spaces $\mathcal{H}_{\Sigma(i,j,k,l)}$ and $\mathcal{H}_{\Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})}$ are isomorphic. In this situation, we can try to characterize locality-preserving isomorphisms between two systems defined on $\Sigma(i, j, k, l)$ and $\Sigma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})$, respectively. This situation is slightly more general than what we considered before (automorphisms of the same system), but it is easy to see that all arguments applied so far extend to this situation. Note that we could have phrased our whole discussion in terms of isomorphisms between different spaces. However, we chose not to do so to minimize the amount of notation required; instead, we only consider this situation in this section. This generalization for the 4-punctured sphere is all we need to treat automorphisms on higher-genus surfaces.

For $\mathcal{H}_{\Sigma(i,j,k,l)}$, we have two bases $\mathcal{B}_C, \mathcal{B}_{C'}$, corresponding to two different DAP-decompositions differing by one loop. Similarly, for $\mathcal{H}_{\Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})}$, we have two bases $\tilde{\mathcal{B}}_C, \tilde{\mathcal{B}}_{C'}$, corresponding to two different DAP-decompositions differing by one loop. We can enumerate the basis elements by the label assigned to this loop. Let $\{\Psi_a\}_a$ and $\{\Psi'_a\}_a$ be the elements of the basis \mathcal{B}_C and $\mathcal{B}_{C'}$, respectively. Here $a \in Q = Q(i, j, k, l) \subset \mathbb{A}$ ranges over the set $Q = Q(i, j, k, l)$ of all elements consistent with the fusion rules, i.e., we must have $\delta_{ija} = \delta_{kla} = 1$. Similarly, let $\{\tilde{\Psi}_{\tilde{a}}\}_{\tilde{a}}$ and $\{\tilde{\Psi}'_{\tilde{a}}\}_{\tilde{a}}$ be the elements of the basis $\tilde{\mathcal{B}}_C$ and $\tilde{\mathcal{B}}_{C'}$, respectively, where now $\tilde{a} \in \tilde{Q} = Q(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})$.

In this situation, we have two basis changes,

$$\Psi'_m = \sum_n \mathbf{V}_{m,n} \Psi_n \text{ where } \mathbf{V}_{m,n} = F_{kln}^{ijm} \quad \text{and} \quad \tilde{\Psi}'_{\tilde{m}} = \sum_{\tilde{n}} \tilde{\mathbf{V}}_{\tilde{m},\tilde{n}} \tilde{\Psi}_{\tilde{n}} \text{ where } \tilde{\mathbf{V}}_{\tilde{m},\tilde{n}} = F_{\tilde{k}\tilde{l}\tilde{n}}^{\tilde{i}\tilde{j}\tilde{m}}.$$

Now consider a locality-preserving isomorphism U which takes the boundary labels (i, j, k, l) to $(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})$. We can then apply the framework above to find the action

$$U\Psi_a = e^{i\varphi_a} \tilde{\Psi}_{\pi(a)} \quad \text{or} \quad U\Psi'_a = e^{i\varphi'_a} \tilde{\Psi}'_{\pi'(a)}$$

on basis states. Here $\pi, \pi' : Q \rightarrow \tilde{Q}$ take fusion-consistent labels (on $\Sigma(i, j, k, l)$) to fusion-consistent labels (on $\Sigma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})$). Because the spaces are isomorphic, we must have $|Q| = |\tilde{Q}|$, hence π, π' can be represented by permutation matrices $\mathbf{\Pi}, \mathbf{\Pi}'$ in the bases $(\{\Psi_a\}_a, \{\tilde{\Psi}_a\}_a)$ or $(\{\Psi'_a\}_a, \{\tilde{\Psi}'_a\}_a)$, respectively. Proceeding similarly with \mathbf{U} , we get the consistency equation $\tilde{\mathbf{V}}\mathbf{U} = \mathbf{U}'\mathbf{V}$ or

$$\tilde{\mathbf{V}}\mathbf{\Pi}\mathbf{D}(\{\varphi_a\}_a) = \mathbf{\Pi}'\mathbf{D}(\{\varphi'_a\}_a)\mathbf{V}, \quad (45)$$

which is expressed in the form of a commutative diagram as in Fig. 9. Equation (45) only differs from equation (38) in allowing boundary labels to change and the basis transformation matrix $\tilde{\mathbf{V}}$ must change accordingly.

For a given set of boundary labels (i, j, k, l) , $(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})$, and a fixed choice of π (which fixes $\mathbf{\Pi}$), any solution $(\mathbf{\Pi}', \{\varphi_a\}_a, \{\varphi'_a\}_a)$ of (45) has phases $\{\varphi_a\}_a$ of the “universal” form

$$\varphi_a = \eta + f(a) \quad \text{for all } a \in Q(i, j, k, l), \quad (46)$$

where $\eta \in [0, 2\pi]$ is an arbitrary global phase independent of a , and f belongs to a set $\text{Iso} \left(\begin{array}{ccc} i & & l \\ j & | & k \end{array} \rightarrow \begin{array}{ccc} \tilde{i} & & \tilde{l} \\ \tilde{j} & | & \tilde{k} \end{array} \right)$ of functions that can be computed from (45) as discussed below.

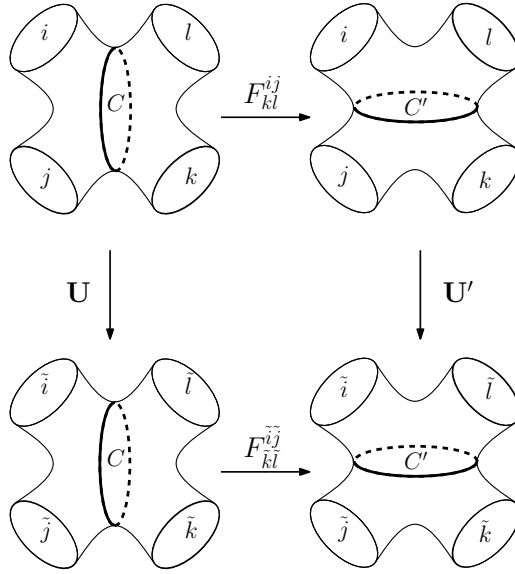


Figure 9: An isomorphism $\mathcal{H}_{\Sigma(i,j,k,l)} \rightarrow \mathcal{H}_{\Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})}$ of two 4-punctured spheres can be given as either \mathbf{U} , which relates the bases \mathcal{B}_C of $\mathcal{H}_{\Sigma(i,j,k,l)}$ to $\tilde{\mathcal{B}}_C$ of $\mathcal{H}_{\Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})}$, or as \mathbf{U}' relating different bases $\mathcal{B}_{C'}$ of $\mathcal{H}_{\Sigma(i,j,k,l)}$ to $\tilde{\mathcal{B}}_{C'}$ of $\mathcal{H}_{\Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})}$. The bases of $\mathcal{H}_{\Sigma(i,j,k,l)}$ and $\mathcal{H}_{\Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})}$ are related through the F -moves F_{kl}^{ij} and $F_{\tilde{k}\tilde{l}}^{\tilde{i}\tilde{j}}$, respectively. The consistency equation (45) can be expressed as a commutative diagram. In the case where $\Sigma(i,j,k,l) = \Sigma(\tilde{i},\tilde{j},\tilde{k},\tilde{l})$ have identical boundary labels such an isomorphism becomes an automorphism, and this reduces to the consistency equation (44).

In summary, we have shown that U acts as

$$U\Psi_a = e^{i\eta}e^{if(a)}\tilde{\Psi}_{\pi(a)} \quad \text{with} \quad f \in \text{Iso} \left(\underline{j \mid i \mid \cdot \mid l \mid k} \rightarrow \underline{\tilde{j} \mid \tilde{i} \mid \pi(\cdot) \mid \tilde{l} \mid \tilde{k}} \right), \quad (47)$$

and where the latter set can be determined by solving the consistency relation (45).

4.4.4 Localization of phases for higher-genus surfaces

We now argue that the phases appearing in Eq. (32) of Proposition 4.2 also factorize into certain essentially local terms, similar to how the overall permutation $\vec{\pi}$ of fusion-consistent labelings decomposes into a collection $\vec{\pi} = \{\pi^C\}_{C \in \mathcal{C}}$ of permutations of labels. More precisely, we will argue that conclusion (47) can be extended to more general surfaces.

Consider a fixed DAP-decomposition \mathcal{C} of Σ . We call a curve $C \in \mathcal{C}$ *internal* if the intersection of Σ with a ball containing C has the form of a 4-punctured sphere with boundary components C_1, C_2, C_3, C_4 consisting of curves ‘neighboring’ C in the DAP decomposition. We call $N(C) = \{C_1, C_2, C_3, C_4\}$ the neighbors (or neighborhood) of C as illustrated in Fig. 10. Key to the following observations is that a basis vector Ψ_ℓ whose restriction to these neighbors is given by $\ell \upharpoonright N(C) = (\ell(C_1), \dots, \ell(C_4))$ gets mapped under U to a vector proportional to $\Psi_{\vec{\pi}(\ell)}$, which assigns the labels $\vec{\pi}(\ell) \upharpoonright N(C) = (\pi^{C_1}(\ell(C_1)), \dots, \pi^{C_4}(\ell(C_4)))$ to the

same curves. This means that the restriction of U to this subspace satisfies similar consistency conditions as the isomorphisms between Hilbert spaces associated with the 4-punctured spheres $\Sigma(\ell \upharpoonright N(C))$ and $\Sigma(\vec{\pi}(\ell) \upharpoonright N(C))$ studied in Section 4.4.2. In particular, for a fixed labeling ℓ the dependence of the phase $\varphi(\ell)$ on the label $\ell(C)$ is given by a function from the set $\text{Iso} \left(\begin{array}{c} i \quad l \\ j \quad | \quad \cdot \quad | \quad k \end{array} \rightarrow \begin{array}{c} \tilde{i} \quad \tilde{l} \\ \tilde{j} \quad | \quad \pi^C(\cdot) \quad | \quad \tilde{k} \end{array} \right)$, where $(i, j, k, l) = \ell \upharpoonright N(C)$ and $(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}) = \vec{\pi}(\ell) \upharpoonright N(C)$. In the following, we simply write $\text{Iso} \left(\ell \upharpoonright N(C) \xrightarrow{\pi^C} \vec{\pi}(\ell) \upharpoonright N(C) \right)$ for this set.

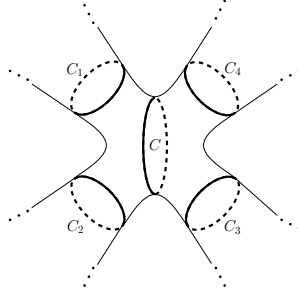


Figure 10: For some DAP-decomposition \mathcal{C} of a surface Σ , a curve $C \in \mathcal{C}$ is considered internal if its neighbors $N(C) = \{C_1, C_2, C_3, C_4\}$ define the boundaries of a 4-punctured sphere.

Proposition 4.4 (Localization of internal phases). *Let U be a locality-preserving automorphism. Let \mathcal{C} be a DAP-decomposition of Σ , and let $\vec{\pi} = \{\pi^C\}_{C \in \mathcal{C}}$ be the family of permutations defined by Proposition 4.1. Let $\varphi(\ell)$ for $\ell \in \mathbf{L}(\mathcal{C})$ be defined by (cf. (32))*

$$U\Psi_\ell = e^{i\varphi(\ell)}\Psi_{\vec{\pi}(\ell)} \quad \text{for all } \ell \in \mathbf{L}(\mathcal{C}) . \quad (48)$$

If $C \in \mathcal{C}$ is internal, then

$$\varphi(\ell) = \eta(\ell \upharpoonright \mathcal{C} \setminus \{C\}) + f_{\vec{\pi} \upharpoonright N(C)}(\ell \upharpoonright N(C), \ell(C)) .$$

for some functions η and f . Furthermore, we have

$$f_{\vec{\pi} \upharpoonright N(C)}(\ell \upharpoonright N(C), \cdot) \in \text{Iso} \left(\ell \upharpoonright N(C) \xrightarrow{\pi^C} \vec{\pi}(\ell) \upharpoonright N(C) \right) .$$

In particular, the dependence of $\varphi(\ell)$ on $\ell(C)$ is “local” and “controlled” by the labeling $\ell \upharpoonright N(C)$ of the neighbors.

In other words, if we fix a family of permutations $\vec{\pi}$, and the labels on the neighbors $N(C)$, then the dependence on the label $\ell(C)$ of the internal edge is essentially fixed.

Proof. We will focus our attention on the subspace $\mathcal{H}_{(i,j,k,l,\star)} \subseteq \mathcal{H}_\Sigma$ spanned by labelings ℓ with $(\ell(C_1), \ell(C_2), \ell(C_3), \ell(C_4)) = (i, j, k, l)$ and $\ell \upharpoonright \mathcal{C} \setminus \{C, C_1, C_2, C_3, C_4\} = \star$ fixed (arbitrarily). For the purpose of this proof, it will be convenient to represent basis vectors Ψ_ℓ associated with such a labeling $\ell \in \mathbf{L}(C)$ as a vector

$$\Psi_\ell = |\ell(C), \ell(C_1), \ell(C_2), \ell(C_3), \ell(C_4), \star\rangle = |a, i, j, k, l, \star\rangle .$$

Defining $\tilde{i} = \pi^{C_1}(i)$, $\tilde{j} = \pi^{C_2}(j)$, $\tilde{k} = \pi^{C_3}(k)$, $\tilde{l} = \pi^{C_4}(l)$, we can rewrite (48) in the form

$$U|a, i, j, k, l, \star\rangle = e^{i\varphi(a, i, j, k, l, \star)} |\pi^C(a), \tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{\star}\rangle,$$

where $\tilde{\star} = \tilde{\pi}_\uparrow(\star)$ for some map $\tilde{\pi}_\uparrow$ taking labelings of the set $\mathcal{C} \setminus \{C, C_1, C_2, C_3, C_4\}$ consistent with (i, j, k, l) to those consistent with $(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})$. We conclude that the restriction of U to $\mathcal{H}_{(i, j, k, l, \star)}$ implements an isomorphism $\mathcal{H}_{(i, j, k, l, \star)} \cong \mathcal{H}_{(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{\star})}$. Since these spaces are isomorphic to $\mathcal{H}_{\Sigma(i, j, k, l)}$ and $\mathcal{H}_{\Sigma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l})}$, respectively, we can apply the result of Section 4.4.3. Indeed, the consistency relation imposed by the F -move is entirely local, not affecting labels associated with curves not belonging to $\{C, C_1, C_2, C_3, C_4\}$. We conclude from (47) that

$$\varphi(a, i, j, k, l, \star) = \eta(i, j, k, l, \star) + f(a), \text{ where } f \in \text{Iso} \left(\underline{j \mid \begin{smallmatrix} i \\ \cdot \end{smallmatrix} \mid l \mid k} \rightarrow \underline{\tilde{j} \mid \begin{smallmatrix} \tilde{i} \\ \pi^C(\cdot) \end{smallmatrix} \mid \tilde{l} \mid \tilde{k}} \right).$$

Since (a, i, j, k, l, \star) were arbitrary, this proves the claim. \square

For example, for $S^2(z^{N+3})$ (as described above), we can apply Proposition 4.4 to the j -th internal edge C_j to obtain

$$\varphi(\vec{x}) = \eta_j(x_1, \dots, \hat{x}_j, \dots, x_N) + f_j(x_{j-1}, x_j, x_{j+1}), \quad (49)$$

where

$$f_j(x_{j-1}, \cdot, x_{j+1}) \in \text{Iso} \left(\underline{x_{j-1} \mid \begin{smallmatrix} z \\ \cdot \end{smallmatrix} \mid x_{j+1}} \rightarrow \underline{\tilde{x}_{j-1} \mid \begin{smallmatrix} z \\ \pi_j(\cdot) \end{smallmatrix} \mid \tilde{x}_{j+1}} \right),$$

and

$$\tilde{x}_{j-1} = \pi_{j-1}(x_{j-1}), \tilde{x}_{j+1} = \pi_{j+1}(x_{j+1}).$$

Here, we use \hat{x}_j to indicate that this argument is omitted.

5 Examples

In what follows, we apply the results of the previous sections to various anyon models. In particular, we show that the local constraint of Proposition 4.1, if specialized to the abelian model $D(\mathbb{Z}_2)$, suffices to imply the result of Theorem 2.11. Surprisingly, this argument does not involve conjugate pairs of loops.

We then show that locality-preserving automorphisms of the Fibonacci model are trivial. For the Ising model, they belong to the Pauli group. More precisely, we will constrain the locality-preserving automorphisms of the $N+3$ -punctured sphere for both models, i.e., the gates acting on the spaces $\mathcal{H}_{S^2(\tau^{N+3})}$ and $\mathcal{H}_{S^2(\sigma^{N+3})}$. In the non-abelian case with $N+3$ anyons of type z , we know from Proposition 4.2 that the action $U|\Psi_\ell\rangle = e^{i\varphi(\ell)}\Psi_{\tilde{\pi}(\ell)}$ on fusion-consistent labelings is parametrized by certain families $\tilde{\pi} = \{\pi^C\}_{C \in \mathcal{C}}$ of permutations, as well as a function φ describing the phase-dependence. To characterize the latter, we

- (i) determine the set of allowed ‘local’ permutations π^C and associated phases f for any occurring internal curve C . This amounts to solving the consistency equation (45) for

the four-punctured sphere, with appropriate boundary labels. For the standard pants decomposition of the $N + 3$ -punctured sphere, this means finding all pairs

$$(\pi_j, f_j) \quad \text{where } f_j \in \text{Iso} \left(\begin{array}{c} z \\ \hline x_{j-1} \end{array} \cdot \begin{array}{c} z \\ \hline x_{j+1} \end{array} \rightarrow \begin{array}{c} z \\ \hline \tilde{x}_{j-1} \end{array} \pi_j(\cdot) \begin{array}{c} z \\ \hline \tilde{x}_{j+1} \end{array} \right) .$$

These correspond to isomorphisms between the Hilbert spaces associated with the labeled surfaces $S^2(z, x_{j-1}, x_{j+1}, z)$ and $S^2(z, \tilde{x}_{j-1}, \tilde{x}_{j+1}, z)$, where $x_{j-1}, \tilde{x}_{j-1} \in Q(j-1), x_{j+1}, \tilde{x}_{j+1} \in Q(j+1)$.

- (ii) we constrain the family $\vec{\pi} = \{\pi^C\}_{C \in \mathcal{C}}$ of allowed permutations by using the global constraints arising from fusion rules and gluing (Proposition 4.3). In the case of $N + 3$ Fibonacci anyons on the sphere with standard pants decomposition \mathcal{C} , dimensional arguments show that all $\pi_j = \text{id}$ are equal to the identity permutation. For Ising anyons, the fusion rules imply that every permutation with even index is equal to the identity permutation, $\pi_{2j} = \text{id}$ (in fact, there is only a single allowed label).
- (iii) we determine the phases $\varphi(\ell)$ by using the localization property of Proposition 4.4 for internal curves C . For $N + 3$ anyons of type z on the sphere, this results in the consistency conditions

$$\varphi(\vec{x}) = \eta_j(x_1, \dots, \hat{x}_j, \dots, x_N) + f_j(x_{j-1}, x_j, x_{j+1}) \quad \text{where} \\ f_j(x_{j-1}, \cdot, x_{j+1}) \in \text{Iso} \left(\begin{array}{c} z \\ \hline x_{j-1} \end{array} \cdot \begin{array}{c} z \\ \hline x_{j+1} \end{array} \rightarrow \begin{array}{c} z \\ \hline \tilde{x}_{j-1} \end{array} \pi_j(\cdot) \begin{array}{c} z \\ \hline \tilde{x}_{j+1} \end{array} \right) \quad \text{for } j = 1, \dots, N .$$

We show that for Fibonacci and Ising anyon models, this system of equations uniquely specifies φ (up to a global phase).

Combining these approaches we obtain the following two results:

Theorem 5.1 (Fibonacci anyon model). *Any locality-preserving automorphism U on the M -punctured sphere $S^2(\tau^M)$ is trivial (i.e., proportional to the identity) if $M \neq 4$. For the four-punctured sphere $S^2(\tau^4)$, $[U]$ belongs (up to a global phase) to the gate set $\{\text{id}, Y\}$, where $Y|\tau\rangle = i|1\rangle$, $Y|1\rangle = -i|\tau\rangle$.*

For the Ising anyon model, the fusion rules imply that $\dim \mathcal{H}_{S^2(z^M)}$ is non-zero only when M is even. In this case, there is a natural isomorphism $\mathcal{H}_{S^2(z^M)} \cong (\mathbb{C}^2)^{\otimes M/2-1}$ (described below, see Eq. (59)). Defining the $(M/2 - 1)$ -qubit Pauli group on the latter space in the usual way, we get the following statement:

Theorem 5.2 (Ising anyon model). *Any locality-preserving automorphism U of $S^2(\sigma^M)$, where $M \geq 1$ is even, belongs to the $(M/2 - 1)$ -qubit Pauli group.*

5.1 The $D(\mathbb{Z}_2)$ model revisited

As a first example, we consider the abelian anyon model $D(\mathbb{Z}_2)$ and rederive a special case of Theorem 2.11. Remarkably, the following argument only involves the local constraint of Proposition 4.1.

Theorem 5.3. *Consider the abelian anyon model $D(\mathbb{Z}_2)$. Then any locality preserving unitary U which preserves the code space has logical action belonging to the homology-preserving Clifford group, i.e., $[U] \in \text{Clifford}_\Sigma^*$.*

Proof. Note that for $D(\mathbb{Z}_2)$, the set of labels is $\mathbb{A} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and in particular $|\mathbb{A}| = 4$. With respect to the ordering $1 = 0 \oplus 0, e = 1 \oplus 0, m = 0 \oplus 1, \epsilon = 1 \oplus 1$, the S -matrix is

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \quad (50)$$

From Proposition 4.1, we immediately get

$$[UP_a(C)U^\dagger] = [P_{\pi^C(a)}(C)] \quad \text{for all } a \in \mathbb{A},$$

where $\pi^C : \mathbb{A} \rightarrow \mathbb{A}$ is a permutation associated with C . In the abelian case where $d_a = 1$ for all $a \in \mathbb{A}$, expression (29) simplifies to

$$\Lambda = S\Pi^{-1}S^{-1}. \quad (51)$$

We claim that for any permutation matrix Π (associated with a permutation $\pi : \mathbb{A} \rightarrow \mathbb{A}$ by $\Pi_{x,y} = \delta_{x,\pi(y)}$), the matrix $\Lambda = \Lambda(\Pi)$ has the form

$$\Lambda = \tilde{\Pi}D, \quad \text{where} \quad \tilde{\Pi} = \tilde{\Pi}(\Pi), D = D(\Pi), \quad (52)$$

for a permutation matrix $\tilde{\Pi}$ and diagonal matrix D with entries ± 1 on the diagonal. According to Proposition 4.1, this implies that $[UF_b U^\dagger] = D_{\pi^{-1}(b), \pi^{-1}(b)} F_{\pi^{-1}(b)}$, showing that U is indeed a homology preserving Clifford group element.

Statement (52) can be checked by direct calculation, considering every (of the 4!) permutation matrices Π . \square

This reasoning applies also, for example, to the double semion model, whose S -matrix is given by

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

However, it is easy to check that the analogous statement of (30) does not hold, e.g., for $D(\mathbb{Z}_3)$, hence this approach does not imply the general statement of Theorem 2.11 for arbitrary abelian models.

5.2 The Fibonacci model

For *Fib*, we have $\mathbb{A} = \{1, \tau\}$ and the only non-trivial fusion rule is $\tau \times \tau = 1 + \tau$ with $d_\tau = \phi = \frac{\sqrt{5}+1}{2}$.

5.2.1 On the torus

The S -matrix (with respect to the ordering $(1, \tau)$) is

$$S = \frac{1}{\sqrt{\phi+2}} \begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix}.$$

The consistency condition (40) for the torus requires that

$$S\Pi\mathbf{D}(\{\varphi_a\}_a)S^{-1} = \Pi'\mathbf{D}(\{\varphi'_a\}_a)$$

where $\Pi, \Pi' \in \left\{ \text{id}_{\mathbb{C}^2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. We consider the two cases:

- For $\Pi = \text{id}$, we get (using $\phi^2 = \phi + 1$)

$$S\Pi\mathbf{D}(\{\varphi_a\}_a)S^{-1} = \frac{1}{2+\phi} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\tau}(1+\phi) & (e^{i\varphi_1} - e^{i\varphi_\tau})\phi \\ (e^{i\varphi_1} - e^{i\varphi_\tau})\phi & e^{i\varphi_1} + e^{i\varphi_\tau}(1+\phi) \end{pmatrix}.$$

For this to be of the form $\Pi'\mathbf{D}(\{\varphi'_a\}_a)$ with a permutation matrix Π' , all entries must have modulus 0 or 1. Since $\phi/(2+\phi) < 1/2$, the off-diagonal elements always have modulus less than 1, and hence must be zero. That is, we must have $e^{i\varphi_1} = e^{i\varphi_\tau} =: e^{i\varphi}$, and it follows that $\Pi' = \text{id}$ and $e^{i\varphi'_1} = e^{i\varphi'_\tau} = e^{i\varphi}$. This implies (according to (39)) that U simply applies a global phase, $U = e^{i\varphi}\text{id}_{\mathbb{C}^2}$.

- For $\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get

$$S\Pi\mathbf{D}(\{\varphi_a\}_a)S^{-1} = \frac{1}{2+\phi} \begin{pmatrix} (e^{i\varphi_1} + e^{i\varphi_\tau})\phi & e^{i\varphi_1}(1+\phi) - e^{i\varphi_\tau} \\ e^{i\varphi_\tau}(1+\phi) - e^{i\varphi_1} & -(e^{i\varphi_1} + e^{i\varphi_\tau})\phi \end{pmatrix}.$$

To have the absolute value of the first entry equal to 0 (it cannot be 1 since $\phi/(2+\phi) < 1/2$, we must have $e^{i\varphi_\tau} = -e^{i\varphi_1}$ (or $\varphi_\tau = \varphi_1 + \pi$) and we get

$$S\Pi\mathbf{D}(\{\varphi_a\}_a)S^{-1} = e^{i\varphi_1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This means that $\Pi' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $(\varphi'_1, \varphi'_\tau) = (\varphi_1 + \pi, \varphi_1)$ is a solution. In summary, we see that (for a global phase $\varphi = \varphi_1$)

$$\begin{aligned} U\Psi_1 &= e^{i\varphi}\Psi_\tau \\ U\Psi_\tau &= -e^{i\varphi}\Psi_1 \end{aligned} \tag{53}$$

is allowed.

Note that this is consistent with the form of a Dehn-twist, given by the logical unitary $U = \text{diag}(1, e^{4\pi i/5})$ (with the ‘topological’ phases or twists on the diagonal), and hence not belonging to the above list: Dehn-twists do not preserve locality (e.g., if it is along C_1 , then an operator supported on C_2 may end up with support in the neighborhood of the union $C_1 \cup C_2$ under conjugation by the realizing unitary).

5.2.2 Fibonacci on the 4-punctured sphere

Before proceeding to the general M -punctured sphere, consider the 4-punctured sphere $S^2(\tau, j, k, \tau)$ with some choice of boundary labels $j, k \in \{1, \tau\}$ satisfying the fusion rules. Our first goal is to characterize locality-preserving isomorphisms between two spaces $\mathcal{H}_{S^2(\tau, j, k, \tau)}$ and $\mathcal{H}_{S^2(\tau, \tilde{j}, \tilde{k}, \tau)}$ in terms of their permutation π and phases f (cf. (43)).

Observe first that if $j = 1$ or $k = 1$, then $\dim \mathcal{H}_{S^2(\tau, j, k, \tau)} = 1$ and we must have $\tilde{j} = 1$ or $\tilde{k} = 1$ for there to be an isomorphism. Furthermore, such an isomorphism is simply a global phase $e^{i\varphi}$. Comparing to (41), we find that the corresponding set of phases can, without loss of generality, be chosen as consisting of the constant 0 function only, $f(1) = f(\tau) = 0$, i.e.,

$$\text{Iso} \left(\underline{j \quad \tau \quad \cdot \quad \tau \quad k} \rightarrow \underline{\tilde{j} \quad \tau \quad \text{id}(\cdot) \quad \tau \quad \tilde{k}} \right) = \{(f(1), f(\tau)) = (0, 0)\}.$$

The only non-trivial case to consider is $\mathcal{H}_{S^2(\tau, \tau, \tau, \tau)}$ (i.e., $j = k = \tilde{j} = \tilde{k} = \tau$) for which $\dim \mathcal{H}_{S^2(\tau, \tau, \tau, \tau)} = 2$. We will take $\{\Psi_1, \Psi_\tau\}$ as an ordered basis. The F -matrix restricted to this subspace in the basis $\{\Psi_1, \Psi_\tau\}$ is

$$F = \frac{1}{\phi} \begin{pmatrix} 1 & \sqrt{\phi} \\ \sqrt{\phi} & -1 \end{pmatrix}.$$

A locality-preserving automorphism U of $\mathcal{H}_{S^2(\tau, \tau, \tau, \tau)}$ has action given by

$$U|\Psi_a\rangle = e^{i\eta} e^{if(a)} |\Psi_{\pi(a)}\rangle \quad \text{where } f \in \text{Iso} \left(\underline{\tau \quad \tau \quad \cdot \quad \tau \quad \tau} \rightarrow \underline{\tau \quad \tau \quad \pi(\cdot) \quad \tau \quad \tau} \right)$$

for some permutation π of $Q(\tau, \tau, \tau, \tau) = \{1, \tau\}$. Let us write id for the identity permutation, and $(\tau, 1)$ for the transposition interchanging τ and 1. Now we find solutions to the consistency relation

$$F\Pi\mathbf{D}(\{\varphi_a\}_a)F^{-1} = \Pi'\mathbf{D}(\{\varphi'_a\}_a)$$

for the two corresponding cases where $\Pi = \text{id}_{\mathbb{C}^2}$ and $\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- For $\Pi = \text{id}_{\mathbb{C}^2}$, we get

$$F\Pi\mathbf{D}(\{\varphi_a\}_a)F^{-1} = \frac{1}{1+\phi} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\tau}\phi & (e^{i\varphi_1} - e^{i\varphi_\tau})\sqrt{\phi} \\ (e^{i\varphi_1} - e^{i\varphi_\tau})\sqrt{\phi} & e^{i\varphi_\tau} + e^{i\varphi_1}\phi \end{pmatrix}.$$

As argued previously in Section 5.2.1 (because $\sqrt{\phi}/(1+\phi) < 1/2$), we conclude any associated unitary U only introduces a global phase. That is, since $\varphi_a = \eta + f(a) = \eta$ implying that $f(a) = 0$ for $a \in \{1, \tau\}$, the only function f in the set is the trivial zero function:

$$\text{Iso} \left(\underline{j \quad \tau \quad \cdot \quad \tau \quad k} \rightarrow \underline{\tilde{j} \quad \tau \quad \text{id}(\cdot) \quad \tau \quad \tilde{k}} \right) = \{(f(1), f(\tau)) = (0, 0)\} \quad (54)$$

- For $\mathbf{\Pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, corresponding to the transposition $(\tau, 1)$, we get

$$F\mathbf{\Pi D}(\{\varphi_a\}_a)F^{-1} = \frac{1}{1+\phi} \begin{pmatrix} (e^{i\varphi_1} + e^{i\varphi_\tau})\sqrt{\phi} & -e^{i\varphi_\tau} + e^{i\varphi_1}\phi \\ -e^{i\varphi_1} + e^{i\varphi_\tau}\phi & -(e^{i\varphi_1} + e^{i\varphi_\tau})\sqrt{\phi} \end{pmatrix}$$

Since $\sqrt{\phi}/(1+\phi) < 1/2$, solutions with $\mathbf{\Pi}' = \text{id}$ are ruled out. Assume that $\mathbf{\Pi}' = \mathbf{\Pi}$. Then the diagonal entries must vanish, implying $e^{i\varphi_\tau} = -e^{i\varphi_1}$ so that there is only a single function f in the set:

$$\text{Iso} \left(\underline{j \quad \tau \quad . \quad \tau \quad k} \rightarrow \underline{\tilde{j} \quad \tau \quad (\tau, 1)(\cdot) \quad \tilde{k}} \right) = \{(f(1), f(\tau)) = (0, \pi)\}$$

Thus,

$$F\mathbf{\Pi D}(\{\varphi_a\}_a)F^{-1} = e^{i\eta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for some global phase η , and the resulting unitary is again of the form (53).

This proves Theorem 5.1 in the special case of the 4-punctured sphere. In other words, the set of locality-preserving gates is non-universal (actually, there is only one non-trivial gate), in sharp contrast to the fact that braiding of τ anyons generates a dense subgroup of $SU(2)$.

5.2.3 M Fibonacci anyons: global constraints from fusion and gluing

Consider now the M -punctured sphere $\Sigma = S^2(\tau^M)$ corresponding to M Fibonacci anyons. We are interested in understanding the spaces $\mathcal{H}_{\Sigma'_j(a,a)}$ for $j \in \{1, \dots, M-3\}$ and $a \in \{1, \tau\}$ (cf. (23)), where Σ'_j is obtained from Σ by cutting along the curve C_j which leaves a $j+2$ -punctured and a $(M-j)$ -punctured sphere, respectively. Note that τ is its own antiparticle: $\tau = \bar{\tau}$, and hence it suffices to consider $\Sigma'_j(\tau, \tau)$ and $\Sigma'_j(1, 1)$. Our goal is to identify pairs (a, \tilde{a}) such that $\mathcal{H}_{\Sigma'_j(a,a)} \cong \mathcal{H}_{\Sigma'_j(\tilde{a}, \tilde{a})}$ are isomorphic, this being a necessary condition for a permutation satisfying $\pi_j(a) = \tilde{a}$ (see Proposition (4.3) and Eq. (36)). To compute $\dim \mathcal{H}_{\Sigma'_j(a,a)}$ for $a \in \{1, \tau\}$, we make use of the general fact that $\dim \mathcal{H}_{S^2(\tau^M)} = \Phi_{M-1}$ where Φ_M denotes the M -th Fibonacci number, starting with $\Phi_0 = 0$ and $\Phi_1 = 1$ and satisfying the recursion relations $\Phi_{M+1} = \Phi_M + \Phi_{M-1}$. From (23), we obtain $\dim \mathcal{H}_{\Sigma'_j(1,1)} = \Phi_j \Phi_{M-j-2}$ and $\dim \mathcal{H}_{\Sigma'_j(\tau, \tau)} = \Phi_{j+1} \Phi_{M-j-1}$, excluding the case $j = 1 = M-3$ which satisfies $\dim \mathcal{H}_{\Sigma'_1(1,1)} = \Phi_1 \Phi_1 = \Phi_2 \Phi_2 = \dim \mathcal{H}_{\Sigma'_1(1,1)}$, it follows from the monotonicity and positivity of Φ that

$$\dim \mathcal{H}_{\Sigma'_j(1,1)} < \dim \mathcal{H}_{\Sigma'_j(\tau, \tau)} \quad \text{for } M > 4, \text{ and all } j \in \{1, \dots, M-3\}. \quad (55)$$

Hence, according to the consistency condition (36), for $M > 4$, we only get an isomorphism $\mathcal{H}_{\Sigma'(a, \tilde{a})} \cong \mathcal{H}_{\Sigma'(\pi^C(a), \pi^C(\tilde{a}))}$ with $\pi^C = \text{id}$ being trivial for any internal loop C in a standard DAP decomposition. However, note that any loop on the surface of the punctured sphere can be found in *some* standard DAP decomposition – namely that formed by splitting the surface along C into two separate punctured spheres (left and right) and ordering the punctures such that those in the left sphere precede those in the right. Using proposition 3.3, we recall that the loop algebras $[\mathcal{A}(C)]$ for all loops generates the full logical algebra $[\mathcal{A}(\Sigma)]$. This therefore fully

specifies the action of any locality-preserving logical unitary $[U]$ as a trivial phase. Note that for the case $M < 4$ any logical unitary is by definition trivial since $\dim S^2(\tau^M) \leq 1$ whereas the case $M = 4$ is less restrictive and has been characterized in section 5.2.2. This completes the proof of Theorem 5.1.

This proof approach emphasizes the stringent conditions imposed by Proposition 4.3 which for many anyon models such as the Fibonacci model, can heavily restrict the logical action of the allowed unitaries. However, this approach crucially relies on the fact that for all internal loops, the permutations of the idempotents are trivial. We now present an illustrative alternative approach reaching the same conclusion.

5.2.4 Localization of internal phases for Fibonacci

In general, it should be possible to fix a basis from a standard DAP decomposition and calculate the consistent permutations π_j for that basis. In the case of the Fibonacci model, having permutations be identity for a standard DAP implies that U is diagonal in the basis $\{\Psi_\ell\}_{\ell \in \mathbf{L}(\mathcal{C})}$ composed of fusion-consistent labelings, i.e.,

$$U\Psi_{\vec{x}} = e^{i\varphi(\vec{x})}\Psi_{\vec{x}} \quad \text{for all } \vec{x} \in \mathbf{L}(\mathcal{C}) .$$

From this point, we may proceed to calculate (or constrain) phases $\varphi(\vec{x})$ associated with that basis. To do so, we may in general use proposition 4.4 as follows. Applying Proposition 4.4 to each internal curve C_j gives

$$\varphi(\vec{x}) = \eta_j(x_1, \dots, \hat{x}_j, \dots, x_N) + f_j(x_{j-1}, x_j, x_{j+1}) ,$$

where

$$f_j(x_{j-1}, \cdot, x_{j+1}) \in \text{Iso} \left(\begin{array}{ccc} \tau & & \tau \\ x_{j-1} \mid & \cdot & \mid x_{j+1} \end{array} \rightarrow \begin{array}{ccc} \tau & & \tau \\ \tilde{x}_{j-1} \mid & \pi_j(\cdot) & \mid \tilde{x}_{j+1} \end{array} \right) .$$

for some functions η_j , $j = 1, \dots, N$. However, since $\pi_j = \text{id}$ for all $j \in \{1, \dots, N\}$ as argued in the previous section, we only have to consider the set (54) which contains just the zero function:

$$\text{Iso} \left(\begin{array}{ccc} \tau & & \tau \\ \tau \mid & \cdot & \mid \tau \end{array} \rightarrow \begin{array}{ccc} \tau & & \tau \\ \tau \mid & \text{id}(\cdot) & \mid \tau \end{array} \right) = \{(f(1), f(\tau)) = (0, 0)\} .$$

That is, $f_j(\tau, \cdot, \tau) = 0$ for all j . We conclude that there are functions $\{\eta_j\}_{j=1}^N$ such that

$$\varphi(\vec{x}) = \eta_j(x_1, \dots, \hat{x}_j, \dots, x_N) \quad \text{for } j = 1, \dots, N . \quad (56)$$

Statement (56) implies that $\varphi(\vec{x}) = \varphi(\vec{x}')$ for any two fusion-consistent labelings $\vec{x}, \vec{x}' \in \mathbf{L}(\mathcal{C})$ that are related by interchanging τ and 1 in a single entry. Since any element in $\mathbf{L}(\mathcal{C})$ can be obtained from the sequence $\tau^N = (\tau, \dots, \tau)$ by such interchanges, this shows that $\varphi(\vec{x}) = \eta$ is a global phase independent of the sequence \vec{x} .

Finally, we can conclude that the only locality-preserving automorphism of $\mathcal{H}_{S^2(\tau^M)}$, for $M \neq 4$, is trivial and implements the identity up to a global phase. This completes the proof of Theorem 5.1.

5.3 The Ising model

The Ising anyon model has label set $\mathbb{A} = \{1, \psi, \sigma\}$ and non-trivial fusion rules

$$\psi \times \psi = 1, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \psi.$$

In what follows, we will use a strategy analogous to the Fibonacci case to determine locality preserving automorphisms U of $\mathcal{H}_{S^2(\sigma^N)}$ corresponding to the $M = N + 3$ -punctured sphere with fixed boundary labels σ .

5.3.1 Ising on the 4-punctured sphere

Consider the possible spaces $\mathcal{H}_{S^2(\sigma, j, k, \sigma)}$ for $\{j, k\} \in \mathbb{A}$, and observe that fusion consistency implies

$$\dim \mathcal{H}_{S^2(\sigma, j, k, \sigma)} = \begin{cases} 0 & \text{if } j \neq k = \sigma \text{ or } k \neq j = \sigma \\ 1 & \text{if } j, k \in \{1, \psi\}, \\ 2 & \text{if } j = k = \sigma. \end{cases}$$

Therefore, the only nontrivial case to consider is $\mathcal{H}_{S^2(\sigma, \sigma, \sigma, \sigma)} = \mathcal{H}_{S^2(\sigma^4)}$ with an ordered basis $\{\Psi_1, \Psi_\psi\}$. A locality-preserving automorphism of $\mathcal{H}_{S^2(\sigma^4)}$ will act as

$$U|\Psi_a\rangle = e^{i\eta} e^{if(a)} |\Psi_{\pi(a)}\rangle \quad \text{where } f \in \text{Iso} \left(\begin{array}{c} \sigma \quad \sigma \\ \sigma \quad \sigma \end{array} \rightarrow \begin{array}{c} \sigma \quad \sigma \\ \sigma \quad \sigma \end{array} \right)$$

A valid permutation π of $\{1, \psi\}$ that defines the action of U , and the set of phases can be determined as follows. Let $\mathcal{B}_{\mathcal{C}} = \{\Psi_1, \Psi_\psi\}$ and $\mathcal{B}_{\mathcal{C}'} = \{\Psi'_1, \Psi'_\psi\}$ be corresponding ordered basis of $\mathcal{H}_{S^2(\sigma^4)}$ for the two DAP-decomposition \mathcal{C} and \mathcal{C}' , respectively. The F -matrix relating these two bases is given in the ordered basis $\mathcal{B}_{\mathcal{C}}$ as

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now consider some locality-preserving automorphism U expressed in the basis $\mathcal{B}_{\mathcal{C}}$ and $\mathcal{B}_{\mathcal{C}'}$ respectively as

$$\mathbf{U} = \mathbf{\Pi} \mathbf{D}(\{e^{i\varphi_\ell}\}_\ell) \quad \text{and} \quad \mathbf{U}' = \mathbf{\Pi}' \mathbf{D}(\{e^{i\varphi'_\ell}\}_\ell),$$

for some 2×2 permutation matrices $\mathbf{\Pi}, \mathbf{\Pi}'$ and diagonal matrices $\mathbf{D}(\{e^{i\varphi_\ell}\}_\ell)$ and $\mathbf{D}(\{e^{i\varphi'_\ell}\}_\ell)$ with corresponding phases. Then the consistency relation takes the form $\mathbf{U}' = F \mathbf{U} F^{-1}$. Next, we find all consistent solutions for a given permutation $\mathbf{\Pi}$.

- For $\mathbf{\Pi} = \text{id}_{\mathbb{C}^2}$, we get

$$F \mathbf{\Pi} \mathbf{D}(\{e^{i\varphi_\ell}\}_\ell) F^{-1} = \frac{1}{2} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\psi} & e^{i\varphi_1} - e^{i\varphi_\psi} \\ e^{i\varphi_1} - e^{i\varphi_\psi} & e^{i\varphi_1} + e^{i\varphi_\psi} \end{pmatrix} = \mathbf{\Pi}' \mathbf{D}(\{e^{i\varphi'_\ell}\}_\ell). \quad (57)$$

Suppose that $\mathbf{\Pi}' = \text{id}_{\mathbb{C}^2}$. Then the consistency relation (57) becomes

$$\frac{1}{2} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\psi} & e^{i\varphi_1} - e^{i\varphi_\psi} \\ e^{i\varphi_1} - e^{i\varphi_\psi} & e^{i\varphi_1} + e^{i\varphi_\psi} \end{pmatrix} = \begin{pmatrix} e^{i\varphi'_1} & 0 \\ 0 & e^{i\varphi'_\psi} \end{pmatrix},$$

which implies $e^{i\varphi_1} = e^{i\varphi_\psi} = e^{i\varphi'_1} = e^{i\varphi'_\psi} =: e^{i\eta}$. Therefore U expressed in the basis \mathcal{B}_C is trivial up to a global phase:

$$\mathbf{U} = e^{i\eta} \cdot \text{id}_{\mathbb{C}^2}.$$

Remaining in the case where $\mathbf{\Pi} = \text{id}_{\mathbb{C}^2}$, suppose instead that $\mathbf{\Pi}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The consistency relation (57) then becomes

$$\frac{1}{2} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\psi} & e^{i\varphi_1} - e^{i\varphi_\psi} \\ e^{i\varphi_1} - e^{i\varphi_\psi} & e^{i\varphi_1} + e^{i\varphi_\psi} \end{pmatrix} = \begin{pmatrix} 0 & e^{i\varphi'_\psi} \\ e^{i\varphi'_1} & 0 \end{pmatrix},$$

which implies $e^{i\varphi_1} = -e^{i\varphi_\psi}$ and $e^{i\varphi'_1} = e^{i\varphi'_\psi} = e^{i\varphi_1}$. Setting $e^{i\eta} := e^{i\varphi_1}$, implies that U expressed in the basis \mathcal{B}_C is given by

$$\mathbf{U} = e^{i\eta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These two solutions of the consistency relation, for the case where $\mathbf{\Pi} = \text{id}_{\mathbb{C}^2}$, now determine the only two functions of the set

$$\text{Iso} \left(\underline{\sigma \mid \sigma} \cdot \underline{\sigma \mid \sigma} \rightarrow \underline{\sigma \mid \text{id}(\cdot)} \mid \underline{\sigma \mid \sigma} \right) = \{(f(1), f(\psi)) = (0, 0), (0, \pi)\},$$

- For $\mathbf{\Pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, corresponding to the transposition $(\psi, 1)$ we get

$$F\mathbf{\Pi}\mathbf{D}(\{e^{i\varphi_\ell}\}_\ell)F^{-1} = \frac{1}{2} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\psi} & e^{i\varphi_1} - e^{i\varphi_\psi} \\ -e^{i\varphi_1} + e^{i\varphi_\psi} & -e^{i\varphi_1} - e^{i\varphi_\psi} \end{pmatrix} = \mathbf{\Pi}'\mathbf{D}(\{e^{i\varphi'_\ell}\}_\ell). \quad (58)$$

By taking $\mathbf{\Pi}' = \text{id}_{\mathbb{C}^2}$, relation (58) becomes

$$\frac{1}{2} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\psi} & e^{i\varphi_1} - e^{i\varphi_\psi} \\ -e^{i\varphi_1} + e^{i\varphi_\psi} & -e^{i\varphi_1} - e^{i\varphi_\psi} \end{pmatrix} = \begin{pmatrix} e^{i\varphi'_1} & 0 \\ 0 & e^{i\varphi'_\psi} \end{pmatrix},$$

which implies $e^{i\varphi_1} = e^{i\varphi_\psi} = e^{i\varphi'_1} = -e^{i\varphi'_\psi}$. Letting $e^{i\eta} := e^{i\varphi_1}$ allows U to be expressed in the basis \mathcal{B}_C by

$$\mathbf{U} = e^{i\eta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Instead, suppose now that $\mathbf{\Pi}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the consistency relation (58) is of the form

$$\frac{1}{2} \begin{pmatrix} e^{i\varphi_1} + e^{i\varphi_\psi} & e^{i\varphi_1} - e^{i\varphi_\psi} \\ -e^{i\varphi_1} + e^{i\varphi_\psi} & -e^{i\varphi_1} - e^{i\varphi_\psi} \end{pmatrix} = \begin{pmatrix} 0 & e^{i\varphi'_\psi} \\ e^{i\varphi'_1} & 0 \end{pmatrix},$$

implying that $e^{i\varphi_1} = -e^{i\varphi_\psi} = e^{i\varphi'_1} = -e^{i\varphi'_\psi}$. Let $e^{i\eta} := e^{i\varphi_1}$, then this shows that U expressed in the basis $\mathcal{B}_\mathcal{C}$ is given by

$$\mathbf{U} = e^{i\eta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, these two solutions completely determine the relevant set of functions (which happens to be the same as the previous case for $\mathbf{\Pi} = \text{id}_{\mathbb{C}^2}$):

$$\text{Iso} \left(\begin{array}{c} \sigma \quad \sigma \\ \sigma \quad | \quad \cdot \quad | \quad \sigma \end{array} \rightarrow \begin{array}{c} \sigma \quad \sigma \\ \sigma \quad | \quad (\psi, 1)(\cdot) \quad | \quad \sigma \end{array} \right) = \{(f(1), f(\psi)) = (0, 0), (0, \pi)\},$$

By denoting the single qubit (logical) Pauli group as

$$\mathcal{P} := \left\{ e^{i\varphi} M : M \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \right\}$$

these results can be summarized as follows: If U is a locality-preserving automorphism of the fusion space $\mathcal{H}_{S^2(\sigma^4)}$ of the 4-punctured sphere, then U expressed in the basis $\mathcal{B}_\mathcal{C}$ is in \mathcal{P} .

5.3.2 M Ising anyons: global constraints from fusion and gluing

Let $M \geq 4$ and consider the $M = N + 3$ -punctured sphere $S^2(\sigma^M)$ and corresponding space $\mathcal{H}_{S^2(\sigma^M)}$. For the ‘standard’ DAP-decomposition \mathcal{C} of $S^2(\sigma^M)$, a consistent labeling $\mathbf{L}(\mathcal{C})$ corresponds to a sequence $(\ell(C_1), \dots, \ell(C_N)) = (x_1, \dots, x_N)$. It is readily observed that $\dim \mathcal{H}_{S^2(\sigma^M)} = 0$ if M is odd, as there are no consistent labelings in this case.

Therefore, in what follows we will restrict our discussion to the $M = N + 3$ -punctured sphere where N is any odd positive integer. In this case, any consistent labeling $\ell \in \mathbf{L}(\mathcal{C})$ yields a sequence (x_1, \dots, x_N) where $x_i \in \{1, \psi\}$ for odd i and $x_i = \sigma$ is fixed for even i . Actually any such labeling of this form is consistent, giving an isomorphism defined in terms of orthonormal basis elements by

$$\begin{aligned} W : \mathcal{H}_{S^2(\sigma^{N+3})} &\rightarrow (\mathbb{C}^2)^{(N+1)/2} \\ |\vec{x}\rangle &\mapsto |x_1\rangle \otimes |x_3\rangle \otimes \dots \otimes |x_N\rangle. \end{aligned} \quad (59)$$

Now consider a locality-preserving automorphism U of $\mathcal{H}_{S^2(\sigma^{N+3})}$ and its associated family $\vec{\pi} = \{\pi_j\}$ of permutations. Because only sequences \vec{x} with $x_{2j} = \sigma$ for all j are fusion-consistent, and $\vec{\pi}$ is a permutation on $\mathbf{L}(\mathcal{C})$, we conclude that $\pi_{2j}(\sigma) = \sigma$ for all j . In other words, we can essentially ignore labels carrying even indices. For odd indices, only labels $x_{2j+1} \in \{1, \psi\}$ are allowed, which means that $\pi_{2j+1} \in \{\text{id}, (\psi, 1)\}$ either leaves the label invariant or interchanges ψ and 1. In conclusion, $\vec{\pi} = \{\pi_j\}_{j=1}^N$ are of the form $\pi_j \in \{\text{id}, (\psi, 1)\}$ for odd j , and $\pi_j = \text{id}$ for even j .

5.3.3 Localization of internal phases for Ising

As argued previously, in this case any even permutation π_{2k} corresponding to a curve C_{2k} with even index cannot change the σ label, i.e., $\pi_{2k}(\sigma) = \sigma$. For odd $j = 2k + 1$, we obtain the constraint

$$\varphi(\vec{x}) = \eta_{2k+1}(x_1, \dots, \widehat{x_{2k+1}}, \dots, x_N) + f_{2k+1}(x_{2k+1}) \quad \text{for } k = 0, \dots, (N-1)/2$$

where $f_{2k+1} \in \text{Iso} \left(\frac{\sigma}{\sigma} \mid \frac{\sigma}{\sigma} \rightarrow \frac{\sigma}{\sigma} \mid \pi_{2k+1}(\cdot) \mid \frac{\sigma}{\sigma} \right)$ given that for even labels $\pi_{2m}(x_{2m}) = x_{2m} = \sigma$. Let us write

$$\varphi(\vec{x}) = \eta(\vec{x}) + \sum_{m=0}^{(N+1)/2} f_{2m+1}(x_{2m+1}) \quad (60)$$

and show that $\eta(\vec{x}) = \eta$ is actually independent of \vec{x} . Indeed, we can write

$$\begin{aligned} \eta(\vec{x}) &= (\varphi(\vec{x}) - f_{2k+1}(x_{2k+1})) - \sum_{m, m \neq k}^{(N+1)/2} f_{2m+1}(x_{2m+1}) \\ &= \eta_{2k+1}(x_1, \dots, \widehat{x_{2k+1}}, \dots, x_N) - \sum_{m, m \neq k}^{(N+1)/2} f_{2m+1}(x_{2m+1}) \end{aligned}$$

Since this holds for all k , we conclude that $\eta(\vec{x}) = \eta(\widehat{x_1}, x_2, \widehat{x_3}, x_4, \dots)$ is a function of the even entries only. But the latter are all fixed as $x_{2m} = \sigma$, hence $\eta(\vec{x}) = \eta$ is simply a global phase.

We can now combine these results into a general statement concerning locality-preserving automorphisms of the M -punctured sphere $S^2(\sigma^M)$. Again, since $\dim \mathcal{H}_{S^2(\sigma^M)} = 0$ for odd M and $\dim \mathcal{H}_{S^2(\sigma^2)} = 1$, we are only concerned with the cases where $M = N + 3 \geq 4$ is even. Let $\{|\vec{x}\rangle\}_{\vec{x} \in \mathcal{L}(\mathcal{C})}$ be a basis of $\mathcal{H}_{S^2(\sigma^M)}$. Then such an automorphism must act on $\mathcal{H}_{S^2(\sigma^M)}$ as

$$U|\vec{x}\rangle = e^{i\varphi(\vec{x})} |\vec{\pi}(\vec{x})\rangle, \quad \text{where} \quad \varphi(\vec{x}) = \eta + \sum_{m=0}^{(N+1)/2} f_{2m+1}(x_{2m+1})$$

and

$$f_{2k+1} \in \text{Iso} \left(\frac{\sigma}{\sigma} \mid \frac{\sigma}{\sigma} \rightarrow \frac{\sigma}{\sigma} \mid \pi_{2k+1}(\cdot) \mid \frac{\sigma}{\sigma} \right) = \{(f(1), f(\psi)) = (0, 0), (0, \pi)\}.$$

More explicitly, we have

$$U|\vec{x}\rangle = e^{i\eta} \left(\prod_{m=1}^{(N+1)/2} e^{if_{2m+1}(x_{2m+1})} \right) |\pi_1(x_1)x_2\pi_3(x_3)x_4 \cdots \pi_N(x_N)\rangle.$$

In particular, under the isomorphism (59), we get

$$WUW^{-1} = e^{i\eta} \bigotimes_{m=1}^{(N+1)/2} U_m \quad \text{where} \quad U_m|x\rangle = e^{if_{2m-1}(x)} |\pi_{2m-1}(x)\rangle.$$

From Section (5.3.1), we know that U_m is a single-qubit Pauli for each m up to a global phase. This concludes the proof of Theorem 5.2.

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References

- [1] M. Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, 68:175–186, 1989.
- [2] S. Beigi, P. W. Shor, and D. Whalen. The quantum double model with boundary: Condensations and symmetries. *Communications in Mathematical Physics*, 306(3):663–694, 2011.
- [3] H. Bombin. Topological order with a twist: Ising anyons from an abelian model. *Phys. Rev. Lett.*, 105:030403, 2010.
- [4] H. Bombin and M. A. Martin-Delgado. Topological quantum distillation. *Phys. Rev. Lett.*, 97, 2006.
- [5] H. Bombin and M. A. Martin-Delgado. Family of non-abelian Kitaev models on a lattice: Topological condensation and confinement. *Phys. Rev. B*, 78:115421, Sep 2008.
- [6] H. Bombin and M.A. Martin-Delgado. Topological computation without braiding. *Phys.Rev.Lett.*, 98:160502, 2007.
- [7] S. Bravyi, M. Hastings, and F. Verstraete. Lieb-Robinson Bounds and the Generation of Correlations and Topological Quantum Order. *Physical Review Letters*, 97(5):050401, July 2006.
- [8] S. Bravyi and A. Y. Kitaev. Quantum codes on a lattice with boundary. 1998. [arXiv:quant-ph/9811052](#).
- [9] S. Bravyi and A. Y. Kitaev. Universal quantum computation with ideal Clifford gates and noisy ancillas. *Phys. Rev. A*, 71:022316, Feb 2005.
- [10] S. Bravyi and R. König. Classification of topologically protected gates for local stabilizer codes. *Phys. Rev. Lett.*, 110:170503, Apr 2013.
- [11] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. *Physical Review B*, 82(15):155138, October 2010.
- [12] B. Eastin and E. Knill. Restrictions on transversal encoded quantum gate sets. *Phys. Rev. Lett.*, 102:110502, Mar 2009.

- [13] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty. Symmetry-protected phases for measurement-based quantum computation. *Phys. Rev. Lett.*, 108:240505, Jun 2012.
- [14] A. G. Fowler, M. Mariantoni, J. M. Martinis, and A. N. Cleland. Surface codes: Towards practical large-scale quantum computation. *Phys. Rev. A*, 86:032324, Sep 2012.
- [15] A. G. Fowler, A. M. Stephens, and P. Groszkowski. High threshold universal quantum computation on the surface code. *Phys. Rev. A*, 80:052312, 2009.
- [16] M. Freedman, C. Nayak, K. Walker, and Z. Wang. *On Picture (2+1)-TQFTs*, chapter 2, pages 19–106. August 2008.
- [17] M. H. Freedman, A. Y. Kitaev, and Z. Wang. Simulation of topological field theories by quantum computers. *Commun. Math. Phys.*, 227:587–603, 2002.
- [18] J. Haah. Local stabilizer codes in three dimensions without string logical operators. *Phys. Rev. A*, 83:042330, 2011.
- [19] J. Haah. An invariant of topologically ordered states under local unitary transformations, July 2014. [arXiv:1407.2926](https://arxiv.org/abs/1407.2926).
- [20] S. P. Humphries. Generators for the mapping class group. In *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, volume 722 of *Lecture Notes in Math.*, pages 44–47. Springer, Berlin, 1979.
- [21] Walker. K. On Witten’s 3-manifold invariants, 1991. Lecture notes, <http://canyon23.net/math/1991TQFTNotes.pdf>.
- [22] A. Kitaev and J. Preskill. Topological entanglement entropy. *Phys. Rev. Lett.*, 96:110404, Mar 2006.
- [23] A. Y. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2, 2003.
- [24] A. Y. Kitaev. Anyons in an exactly solved model and beyond. *Ann. Phys.*, 321(1):2, 2006.
- [25] A. Y. Kitaev and L. Kong. Models for gapped boundaries and domain walls. *Communications in Mathematical Physics*, 313(2):351–373, 2012.
- [26] L. Kong and X.-G. Wen. Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions, May 2014. [arXiv:1405.5858](https://arxiv.org/abs/1405.5858).
- [27] S. M. Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, 1998.
- [28] M. A. Levin and X.-G. Wen. String-net condensation: A physical mechanism for topological phases. *Phys. Rev. B*, 71:045110, Jan 2005.
- [29] M. A. Levin and X.-G. Wen. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.*, 96:110405, Mar 2006.

- [30] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Proc. Cambridge Philos. Soc.*, 60:769–778, 1964.
- [31] Elliott H. Lieb and Derek W. Robinson. The finite group velocity of quantum spin systems. *Communications in Mathematical Physics*, 28(3):251–257, 1972.
- [32] K. Michnicki. 3-d quantum stabilizer codes with a power law energy barrier. 2012. [arXiv:1208.3496](https://arxiv.org/abs/1208.3496).
- [33] G. Moore and N. Seiberg. Polynomial equations for rational conformal field theories. *Physics Letters B*, 212(4):451–460, October 1998.
- [34] F. Pastawski and B. Yoshida. Fault-tolerant logical gates in quantum error-correcting codes, August 2014. [arXiv:1408.1720](https://arxiv.org/abs/1408.1720).
- [35] J. Preskill. Lecture notes on quantum computation, 2004. available at <http://www.theory.caltech.edu/people/preskill/ph229/> lecture.
- [36] G. Segal. *The definition of conformal field theory*, volume 308. London Math. Soc. Lecture Note Ser., Cambridge University Press, 2004. preprint.
- [37] E. Verlinde. Fusion rules and modular transformations in 2d conformal field theory. *Nucl. Phys. B*, 300:360–376, 1988.
- [38] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.